

Regimes of classical transport of cold gases in a two-dimensional anisotropic disorder

L. Pezzé, M. Robert-de-Saint-Vincent*, T. Bourdel, J.-P. Brantut†,
B. Allard, T. Plisson, A. Aspect, P. Bouyer, and L. Sanchez-Palencia
*Laboratoire Charles Fabry de l'Institut d'Optique, CNRS and Univ. Paris-Sud,
campus Polytechnique RD128, F-91127 Palaiseau cedex, France*

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We numerically study the dynamics of cold atoms in a two-dimensional disordered potential. We consider an anisotropic speckle potential and focus on the classical dynamics, which is relevant to some recent experiments. Firstly, we study the behavior of particles with a fixed energy and identify different transport regimes. For low energy, the particles are classically localized due to the absence of a percolating cluster. For high energy, the particles undergo normal diffusion and we show that the diffusion coefficients scale algebraically with the particle energy, with an anisotropy factor which significantly differs from that of the disordered potential. For intermediate energy, we find a transient sub-diffusive regime, which is relevant to the time scale of typical experiments. Secondly, we study the behavior of a cold-atomic gas with an arbitrary energy distribution, using the above results as a groundwork. We show that the density profile of the atomic cloud in the diffusion regime is strongly peaked and, in particular, that it is not Gaussian. Its behavior at large distances allows us to extract the energy-dependent diffusion coefficients from experimental density distributions. For a thermal cloud released into the disordered potential, we show that our numerical predictions are in agreement with experimental findings. Not only does this work give insights to recent experimental results, but it may also serve interpretation of future experiments searching for deviation from classical diffusion and traces of Anderson localization.

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I. INTRODUCTION

Transport processes are widespread and common in many fields of physics, chemistry and biology and have drawn the attention of both theorists and experimentalists. Diffusion of waves and particles in disordered media determines the behaviors of a variety of physical systems. These include electronic conductivity in dirty materials [1], superfluidity of liquid helium in porous substrates [2], and diffusion of light in dense media [3], with applications to dielectric materials [4] or interstellar clouds [5]. Diffusion is at the heart of transport phenomena in most systems of condensed-matter physics. For instance, it underlies the Drude theory of conductivity [6], as well as the self-consistent theory of Anderson localization [7]. A major hindrance to the complete understanding of condensed-matter systems is the complex interplay of many relevant ingredients, such as the structure and thermal fluctuations of the underlying substrate [6], the disorder [8], and the inter-particle interactions, which can lead either to superconductivity [9] or to metal-insulator transitions [10, 11]. This poses a number of fundamental questions regarding the existence and the nature of disorder-induced metal-insulator transitions [12], the role of the spatial dimension and the effect of non-linearities. Currently, much attention is devoted to disordered quantum gases [13–17]. These systems offer unprecedented control of most parameters, including temperature, geometry and particle-particle interaction [18–21]. Moreover, disordered potentials can be engineered almost at will, and their statistical properties can be controlled and tuned [22]. So far, a large body of theoretical work has been devoted to Anderson localiza-

*Current address: Physikalisches Institut der Universität Heidelberg Philosophenweg 12, 69120 Heidelberg, Germany

†Current address: Institute for Quantum Electronics, ETH Zürich, Hüngrberg, CH-8093 Zürich, Switzerland

tion of non-interacting particles [23–35], interacting Bose [36–47] and Fermi [48–51] disordered gases, and two-component disordered systems [52–58]. Experimentally, classical suppression of transport of Bose-Einstein condensates [59, 60] and Anderson localization of matter-waves [61, 62] in one dimension (1D) have been reported recently. The competition between interaction and disorder has been investigated in 1D bi-chromatic optical lattices [63, 64] and in three dimensional (3D) disordered lattices [65, 66].

Studying the dynamics of even non-interacting atoms in a disordered potential is a difficult problem, especially in dimensions higher than one. For classical particles, the motion in disordered potentials is generally chaotic, which sparks a variety of dynamical regimes [67], including classical localization, normal diffusion as well as sub- and super-diffusion. For waves, diffusion is obtained in the limit of incoherent multiple scatterings [7, 68, 75, 76]. Multiple coherent scatterings [68–70] from the random defects can induce weak [72] or strong [29] localization effects. With a view towards search for localization effects in cold-atomic gases, the characterization of classical transport regimes is a central task. It is of special interest in dimension two (2D), which is the marginal dimension for return probability in Brownian motion and for Anderson localization [71]. In the context of cold-atomic gases, diffusion has been experimentally studied in random dissipative 3D optical molasses [73, 74], and conservative 2D random potentials [77].

In this work, we numerically study the transport properties of cold atoms in 2D disorder. We consider a speckle potential and focus on the classical regime such that

$$\lambda_{\text{dB}} \ll \sigma_{\text{R}} \leq l_{\text{B}} \ll L \ll L_{\text{loc}}, \quad (1)$$

where σ_{R} is the characteristic length scale of the disordered potential, λ_{dB} is the atomic de Broglie wavelength, l_{B} is the Boltzmann (transport) mean free path, L is the system size and L_{loc} is the localization length. The inequality $\lambda_{\text{dB}} \ll l_{\text{B}}$ guarantees that wave interference effects, which are important in 2D, occur on a very large scale which will be assumed here to be much larger than the system size and thus disregarded. In particular, besides that, in 2D, all quantum states are localized [71], the inequality $L \ll L_{\text{loc}}$ allows us to neglect strong localization effects. When neglecting wave interference effects, the main dynamical regime is provided by diffusion [3, 7]. For $\sigma_{\text{R}} \leq l_{\text{B}}$ (which is most often true), the condition $\sigma_{\text{R}} \lesssim \lambda_{\text{dB}}$ may be fulfilled. In this case, the wave nature of the particles governs the scattering from each defect. Here instead we consider the opposite condition $\lambda_{\text{dB}} \ll \sigma_{\text{R}}$ meaning that the atoms scatter in the disordered potential as purely classical particles, hence satisfying the Newton equations of motion. Finally, since l_{B} is the typical length of ballistic trajectories, the inequality $l_{\text{B}} \ll L$ ensures that the number of scattering events in the system is sufficiently large to strongly affect the motion of particles. As discussed in our paper, the regime indicated by Eq. (1) is relevant to some recent cold-atom

experiments [77].

The central aim of our paper is to analyze the transport regimes of classical particles at different energy. In particular, we show that they strongly depend on the topographic properties of the disordered potential. An additional aim is to use these results to determine the behavior of atomic clouds with a broad energy distribution, as relevant for cold-atom experiments.

A. Main results presented in the paper.

Here, we give a general overview of the main results of this manuscript (leaving details for the text). Figure 1 illustrates the results obtained for an anisotropic speckle potential of anisotropy factor $\lambda = \sigma_x/\sigma_y$, with σ_r the correlation length in the $r \in \{x, y\}$ direction. At any energy, a particle first undergoes a ballistic flight for a typical time $\tau^* \sim l_{\text{B}}/v$ where v is its initial velocity. At sufficiently small energy, E , all classically allowed regions in the disordered potential have a finite size [see Fig. 1(a)], and the particle remains classically localized in the one containing its initial position. For a particle initially at the origin, the disorder-averaged mean square displacement, $\langle r(t)^2 \rangle \propto t^{\alpha_r}$, shows a sub-diffusive behavior ($\alpha_r < 1$) at intermediate times, which corresponds to the random exploration by the particle of its classically allowed region. Once the full region is explored, $\langle r(t)^2 \rangle$ saturates asymptotically in time ($\alpha_r = 0$) to a finite value [see Fig. 1(d)]. Being of purely topographical nature, this behavior leads to $\langle x^2 \rangle / \langle y^2 \rangle = \lambda^2$. A critical energy E_c signals the percolation phase transition corresponding to the appearance of infinitely extended allowed regions. For E slightly above E_c , the percolating cluster (classically-allowed region) is characterized by large lakes connected by narrow bottlenecks [see Fig. 1(b)], which are located at the saddle points of the disordered potential. The particle then spends long times exploring a lake, with a similar dynamics as described above, leading to the sub-diffusive behavior observed in Fig. 1(e) at intermediate (but experimentally relevant) times. For longer times and larger distances, the dynamics is dominated by rare but possible transitions between the lakes, which leads, asymptotically in time, to normal diffusion, $\langle r(t)^2 \rangle = 2D_r t$. At larger energy, the lakes merge and the size of forbidden regions rapidly decreases [see Fig. 1(c)]. Then, the dynamics is characterized by normal diffusion for $t > \tau^*$ [see Fig. 1(f)]. In this regime, we identify a power-law behavior of the diffusion coefficients as a function of the particle energy and show that the anisotropy of diffusion significantly differs from the anisotropy of the disorder, $D_x/D_y \neq \lambda^2$.

It is worth noting that a sub-diffusion regime should not be viewed as a precursor of localization. In fact, we see in the example considered here that sub-diffusion can cross over to either localization (for $E \lesssim E_c$) or normal diffusion (for $E \gtrsim E_c$).

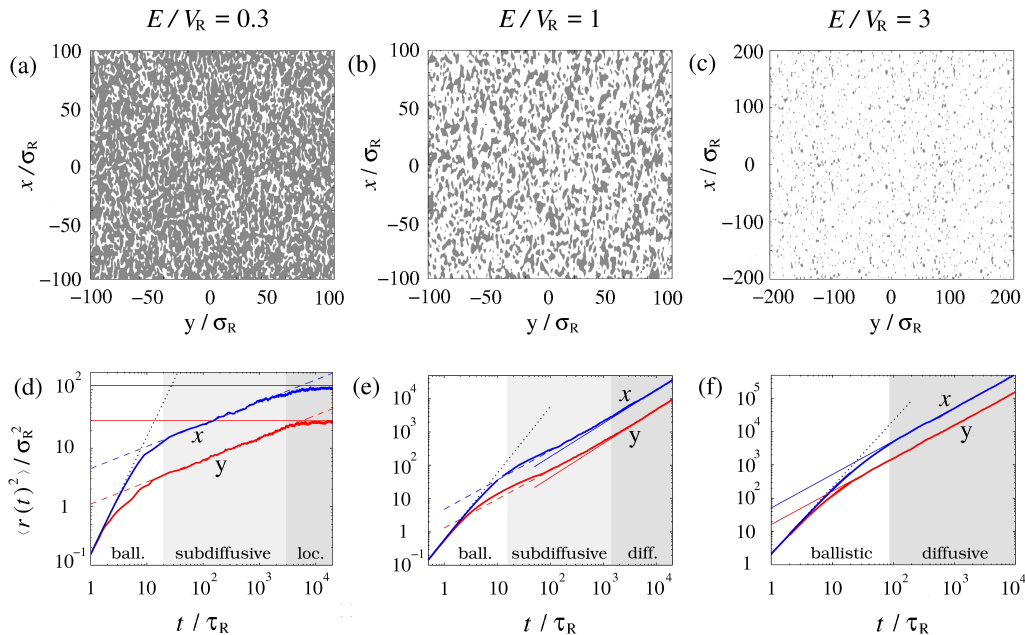


Figure 1: (color online) Topography and transport regimes of classical particles in a 2D, anisotropic, blue-detuned speckle potential with the anisotropy factor $\lambda = \sigma_x/\sigma_y = 2$ (see Sec. II A). Panels (a)-(c) show the allowed (white; $V(\mathbf{r}) \leq E$) and forbidden (black; $V(\mathbf{r}) > E$) regions for a classical particle of energy E in the disordered potential $V(\mathbf{r})$. Different columns correspond to the same realization of the disordered potential and different energies: (a) and (d) $E = 0.3V_R$; (b) and (e) $E = 1V_R$; (c) and (f) $E = 3V_R$, where V_R is the average amplitude of the disordered potential (see Sec. II A). In panel (a), the energy is below the percolation threshold $E_c = 0.52V_R$ (for the 2D blue-detuned speckle potential used here, see Sec. II B) and all the particles are classically localized. In panels (b) and (c), the energy is above the percolation threshold and there appear allowed regions connecting the opposite sides of the box. Panels (d)-(f) show the mean-square displacement, $\langle r(t)^2 \rangle$ (the direction $r = x$ is represented by the upper thick blue line, $r = y$ by the lower thick red line) as a function of time in units of $\tau_R \equiv \sqrt{m\sigma_R^2/|V_R|}$, where m is the mass of the atoms (see Sec. III A). The black dotted line is the isotropic ballistic behavior, $\langle r(t)^2 \rangle \propto t^2$ predicted by Eq. (14). The dashed lines are $\langle r(t)^2 \rangle \propto t^{\alpha_r}$ [see Eq. (13)] for intermediate time scales giving a sub-diffusive behavior ($\alpha_r < 1$) in (d) and (e). The solid lines are $\langle r(t)^2 \rangle \propto t^{\alpha_r}$ for long times giving a time independent behavior ($\alpha_r = 0$) in (d) and a diffusive scaling ($\alpha_r = 1$) in (e) and (f).

B. Outlook.

The manuscript is organized as follows. In Sec. II, we acquaint with the main statistical and topographic properties of the 2D speckle potential used along the paper. We identify a percolation phase transition and determine the critical exponent characterizing the divergence of the classical localization length. In Sec. III, we analyze the dynamics of non-interacting classical particles of given energy in the disordered potential. Firstly, we write down the classical equations of motion and, with a proper rescaling, we highlight the universality class of the classical dynamics. Due to the redistribution of kinetic and potential energy induced by the disordered potential, the total particle energy is identified as the sole relevant parameter for the dynamics. Different transport regimes are then identified and discussed: (i) classical localization, (ii) normal diffusion and (iii) transient subdiffusion. Secondly, in Sec. IV, we use the above results as a groundwork to study the overall dynamics of atomic gases with a broad energy distribution, as relevant to cold-

atom experiments. In particular, we show that, in the diffusion regime, the mean square size of the gas grows linearly in time ($\langle r(t)^2 \rangle \propto t$) but, due to the summation of all diffusive components with energy-dependent diffusion coefficients, expanding density profiles significantly deviate from Gaussian functions. In Sec. V, considering a thermal cloud released in the disordered potential, we compare our predictions with the experimental results of Ref. [77]. We provide fitting functions to extract the single-energy diffusion coefficients from the collective expansion of an atomic cloud of atoms. We find fair agreement between numerical calculations and experimental measurements. Finally, we summarize our findings and discuss possible extensions of this work in Sec. VI. On the one hand, our analysis provides a guideline for the experimental investigation of classical transport in cold atomic gas in disordered potentials [77]. On the other hand, it may contribute to the interpretation of on-going experiments searching for deviation from classical diffusion and traces of Anderson localization with ultracold gases in 2D geometry.

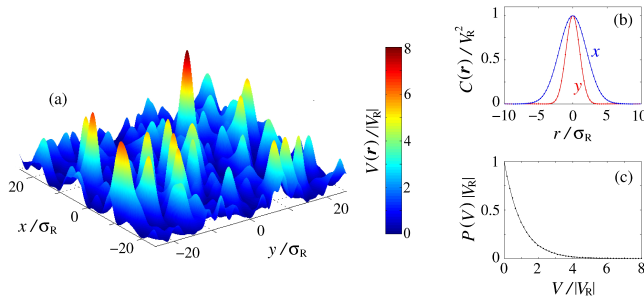


Figure 2: (color online) (a) Example of anisotropic speckle potential used in the numerical simulations. The anisotropy factor is $\lambda = \sigma_x/\sigma_y = 2$. (b) Autocorrelation function of the disordered potential (dots), along the x and y directions [$C(x, y = 0)$ and $C(x = 0, y)$, respectively]. It has been obtained numerically by averaging over 100000 realizations of the disordered potential. The solid lines correspond to Eq. (3) with no fitting parameters. (c) Intensity distribution obtained for a single realization of the disordered speckle potential (dots). The solid line is Eq. (2).

II. DISORDERED POTENTIAL

A. Statistical properties of the speckle potential

Throughout the paper, we consider a speckle disorder, as commonly devised in cold-atom experiments [22, 59–61, 77–80]. A speckle pattern is created, for instance, by a coherent monochromatic laser beam passing through a diffusive plate [22]. A fully developed speckle field can be produced by a ground glass whose surface contains random grains associated to optical path length fluctuations longer than the laser wavelength. In this case, the speckle field, observed at the focal plane of a converging lens, results from the coherent superposition of waves originating from different scattering sites on the rough surface with uncorrelated phases uniformly distributed in $[0, 2\pi]$. Owing to the central limit theorem, the real and imaginary parts of the electric field are independent Gaussian variables. The light-shift potential $V(\mathbf{r})$ felt by the atoms exposed to the electromagnetic radiation is proportional to the field intensity: it is thus a random potential but we stress that its statistics is not Gaussian. The speckle potential is also inversely proportional to the detuning of the laser light with respect to the two-level atomic transition and can be either attractive or repulsive (see below). Finally, assuming a sufficiently far-detuned laser field, we will neglect, in the following, dissipative effects.

The statistical properties of the speckle potential have been extensively investigated [81]. The probability that the potential, at a given point \mathbf{r} , has amplitude $V \equiv V(\mathbf{r})$ follows an exponential law

$$P(V) = \frac{e^{-V/V_R}}{|V_R|} \Theta\left(\frac{V}{V_R}\right), \quad (2)$$

where Θ is the Heaviside step function. We have

$\langle V(\mathbf{r}) \rangle = V_R$ and $(\Delta V(\mathbf{r}))^2 \equiv \langle V(\mathbf{r})^2 \rangle - \langle V(\mathbf{r}) \rangle^2 = V_R^2$, where $\langle \dots \rangle$ indicates the average over the different realizations of the disordered potential. The quantity V_R is the amplitude of the disorder, and its sign depends on the detuning of the laser light from the atomic resonance: For a blue-detuned speckle potential, the disorder is repulsive (i.e. positive $V_R > 0$ and $V(\mathbf{r}) \geq 0$) and it is characterized by arbitrary large intensity peaks with exponentially low probability. Conversely, for a red-detuned speckle potential, the disorder is attractive (i.e. negative $V_R < 0$ and $V(\mathbf{r}) \leq 0$) and it is characterized by arbitrary low dips with exponentially low probability.

The autocorrelation function of the disordered potential, $C(\mathbf{r}) = \langle V(\mathbf{r} + \mathbf{r}')V(\mathbf{r}') \rangle - \langle V(\mathbf{r}') \rangle^2$, can be controlled by the geometry of the diffusive plate [22]. Without loss of generality, it can be written as $C(\mathbf{r}) = V_R^2 c(\mathbf{r}/\sigma_R)$ [82], where σ_R is the correlation length of the disordered potential, that is the typical width of $C(\mathbf{r})$, which is proportional to the speckle grain size. The precise definition of σ_R depends on the specific model of disorder. In this manuscript, we will focus on a 2D, anisotropic, speckle potential with a Gaussian correlation function of widths σ_x and σ_y along the two main directions,

$$C(\mathbf{r}) = V_R^2 \exp(-x^2/2\sigma_x^2) \exp(-y^2/2\sigma_y^2). \quad (3)$$

We define $\sigma_R \equiv \sigma_x$ and $\lambda \equiv \sigma_x/\sigma_y$ the anisotropy factor. In Fig. 2(a), we show a typical realization of a speckle potential as used in the numerical simulations ($\lambda = 2$). Figures 2(b) and 2(c) show its autocorrelation function and intensity distribution, respectively. Experimentally, a speckle potential with a Gaussian correlation function can be obtained by illuminating a diffusive plate with a Gaussian laser beam. The anisotropy can be controlled by the aperture function of the diffusive plate [22, 81] or by tilting a 2D, isotropic speckle field with respect to the diffusion plane of the atoms [77].

B. Topographic properties of the speckle potential

We now study some topographic properties of the 2D speckle potential which are relevant to understand the transport regimes of classical particles [see Sec. III]. Some of them, as, for instance, the percolation threshold, which will be discussed below, do not depend on the anisotropy of the potential. Therefore, these are straightforwardly obtained from those of the isotropic case, upon rescaling of the y direction by the anisotropy factor λ . We thus focus, without loss of generality, on an isotropic speckle disorder in this sub-section.

The positivity of the kinetic energy constraints the trajectories of classical particles with total energy E to the sub-space defined by $V(\mathbf{r}) \leq E$. This condition separates the space in allowed and forbidden regions. For the sake of illustration, Figs. 1(a)-(c) show the classically-allowed ($V(\mathbf{r}) \leq E$; white) and forbidden ($V(\mathbf{r}) > E$; black) regions for particles of fixed energy in a specific realization of a 2D blue-detuned anisotropic speckle potential.

In a continuous disordered potential, the topography of classically allowed regions strongly depends on the ratio $E/|V_R|$, and, for a speckle potential, on the sign of V_R . The fraction of space satisfying the condition $V(\mathbf{r}) \leq E$ is given by [83]

$$\phi(E) = \int_{-\infty}^E dV P(V). \quad (4)$$

For a speckle potential, $P(V)$ is given by Eq. (2). For blue-detuned speckle potentials, we thus find $\phi(E) = 1 - e^{-E/|V_R|}$ for $E \geq 0$ and $\phi(E) = 0$ (i.e. there is no allowed region) for $E \leq 0$. For red-detuned speckle potentials, we have $\phi(E) = e^{+E/|V_R|}$ for $E \leq 0$ and $\phi(E) = 1$ (i.e. there are no forbidden regions) for $E \geq 0$.

For 2D disordered potentials, there exists a critical energy, the so-called *percolation threshold*, E_c , which marks the sharp transition between the existence of infinitely extended allowed and forbidden regions [83]. The *percolation transition* is often considered as the simplest possible phase transition with nontrivial critical behavior [84]. It has a pure geometrical nature and exhibits long-range correlation near the critical value, corresponding to the divergence of the average size of the allowed regions (in infinite space).

Below E_c , the allowed regions are disconnected and form “lakes” of finite size [see Fig. 1(a)]. The allowed regions consist of a finite number of minima of the potential which are connected through saddle points. In other words, particles of energy $E \leq E_c$ are classically localized, i.e. they cannot move to arbitrary large distances. On a macroscopic scale, the system behaves as an insulator. Above E_c , an infinite number of “lakes” merge together to form an infinite “ocean”: The allowed subspace contains at least one connected subset (cluster) which stretches to infinity [see Figs. 1(b)-(c)]. In other words, particles with energy $E > E_c$ and initially placed in an infinitely-extended allowed region can move to infinity. The system thus behaves as a conductor.

A challenging issue is to determine the value of the percolation threshold E_c for different models of disorder. In the case of a disordered potential with sign symmetry, [i.e. when the statistical properties of the potential $V(\mathbf{r})$ are equivalent to those of $2\langle V(\mathbf{r}) \rangle - V(\mathbf{r})$], the percolation threshold is $E_c = \langle V(\mathbf{r}) \rangle$ and the corresponding critical area fraction is $\phi_c \equiv \phi(E_c) = 1/2$ [83]. An example is given by Gaussian random distributions. However, a speckle potential is non-Gaussian and does not have the sign symmetry discussed above. For instance, a blue-detuned speckle potential is bounded below by zero, but it is unbounded above (and *vice versa* for the red-detuned speckle potential). We have numerically investigated the percolation threshold in 2D speckle potentials with the correlation function given by Eq. (3) with $\lambda = 1$. In particular, we have evaluated the probability (for different realizations of the disorder), P_{perc} , to have an allowed region connecting the opposite sides of a square box. The results are shown in Fig. 3 for a blue-detuned speckle and for different box lengths, L . As expected,

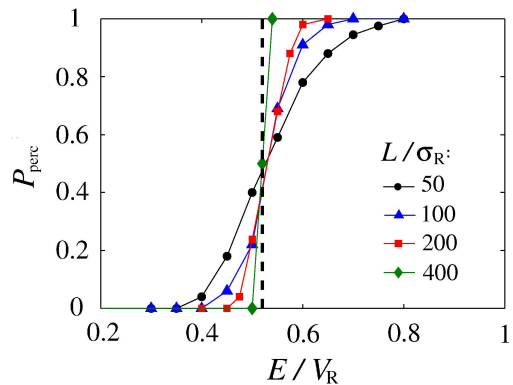


Figure 3: (color online) Percolation probability as a function of the particle energy. It is calculated numerically as the probability to find at least one allowed region connecting the opposite sides of a square box of size L in a blue-detuned (isotropic) speckle potential. Different symbols correspond to different values of L (indicated in the figure) and the solid lines are guides to the eye. The vertical dashed line indicates the percolation threshold, $E_c = 0.52V_R$. The percolation probability is calculated over 100 realizations of the disorder and the numerical grid step is $\sigma_R/10$.

the crossover between $P_{\text{perc}} = 0$ and $P_{\text{perc}} = 1$ becomes sharper and sharper by increasing L , and all curves cross the value $P_{\text{perc}} = 0.5$ at approximately the same value of E/V_R . This is compatible with a phase transition at $E_c = 0.52V_R$, for a blue-detuned, 2D, speckle potential, in the limit $L \rightarrow \infty$ (vertical dashed line in the figure). According to Eq. (4), the critical fraction of allowed regions is given by $\phi_c = 0.405$, a value significantly different from 0.5 as expected for potentials with sign-symmetry. The values $E_c = 0.52V_R$ and $\phi_c = 0.405$ found here are in agreement with the experimental findings [85] and numerical results [86] obtained by mapping the speckle potential onto a network connecting potential minima through saddle points. Because of the duality between allowed and forbidden regions in the 2D case, when inverting the sign of the potential, allowed regions for a particle of energy E becomes forbidden regions for a particle of energy $-E$. The percolation threshold for the red-detuned speckle potential can then be directly calculated from that of the blue-detuned speckle potential: $E_c(-V_R) = -E_c(V_R)$. For red-detuned speckle potentials, we thus find that the percolation threshold is at $E_c = -0.52|V_R|$ and the corresponding fraction of allowed regions is $\phi_c = 0.595$.

In order to further characterize the topographic properties of the speckle potential in a homogeneous box we have studied the classical localization length $\mathcal{L}(E)$, i.e. the mean diameter of the percolating cluster stretching from the box center. We use the definition

$$\mathcal{L}(E) \equiv 2\sqrt{\delta \mathbf{r}^2}, \quad (5)$$

where $\delta \mathbf{r}^2 = \int d^2 \mathbf{r} A_E(\mathbf{r}) |\mathbf{r}|^2 / \int d^2 \mathbf{r} A_E(\mathbf{r})$ with $A_E(\mathbf{r}) = 1$ in the classically allowed region stretching from the

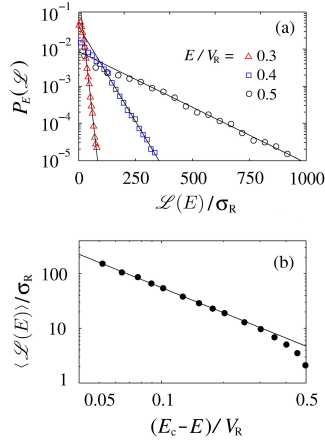


Figure 4: (color online) (a) Probability distribution of the classical localization length defined in Eq. (5), for a blue-detuned speckle potential. Symbols are numerical results for different energies (indicated in the figure). The solid lines are fits to an exponential function. For instance, for $E = 0.3V_R$ (triangles), the fit gives $P_E(\mathcal{L}) \simeq (0.35/\sigma_R) \exp(-0.12 \mathcal{L}/\sigma_R)$ for $\mathcal{L} \gtrsim 30\sigma_R$. (b) Average localization length as a function of $E_c - E$. The line is a fit to Eq. (6) where a , E_c and ν_c are the fitting parameters. The numerical results (dots) have been obtained after averaging over 10000 realizations of a blue-detuned isotropic speckle potential in a box of length $L = 800\sigma_R$ with grid step $\sigma_R/8$.

box center [if $V(\mathbf{r} = 0) \leq E$] and $A_E(\mathbf{r}) = 0$ elsewhere. The calculation of the localization length, Eq. (5), at fixed energy is repeated for several realizations of the disordered potential. The corresponding probability distribution $P_E(\mathcal{L})$ is shown in Fig. 4(a), for various energies. It decreases exponentially for sufficiently large values of \mathcal{L} . The average localization length, $\langle \mathcal{L}(E) \rangle \equiv \int d\mathcal{L} \mathcal{L} P_E(\mathcal{L})$ is shown in Fig. 4(b). It diverges when approaching the percolation threshold E_c and, close to this value, it can be well fitted to the function

$$\langle \mathcal{L}(E) \rangle = a \left(\frac{E_c - E}{V_R} \right)^{-\nu_c}, \quad (6)$$

where $a = 1.63 \pm 0.3 \sigma_R$, the critical exponent $\nu_c = 1.53 \pm 0.2$ and the percolation threshold $E_c = 0.54 \pm 0.02$ are fitting parameters. The value of E_c obtained from the fit is in agreement, within the numerical error, with the value $E_c = 0.52V_R$ found with other methods [see Fig. 3].

Above the percolation threshold, for $E > E_c$, not all the allowed energy regions extend to infinity and some particles can still be classically localized, depending on their initial position. This behavior is relevant to our system, where the dynamics starts *inside* the disordered potential. We have thus studied, as a function of the particle energy, the probability P_R to find, for different realizations of the disordered potential, an allowed region connecting two points at a distance $R = \sqrt{x^2 + y^2}$. The results are shown in Figs. 5(a) and (b) for the blue- and

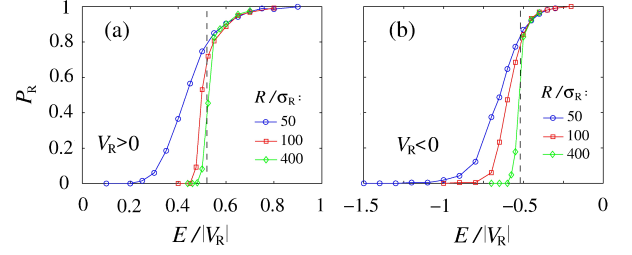


Figure 5: (color online) Probability to find a classically-allowed region stretching to a distance larger than R . Each point has been obtained after averaging over 500 realizations of the disordered potential. Different lines refer to different values of R (indicated in the figure). Panels (a) and (b) correspond to blue- and red-detuned speckle potentials, respectively. The vertical dashed lines are the percolation threshold $E_c = 0.52V_R$.

red-detuned speckle potentials, respectively. For large enough distance R , the probability shows a sharp transition, from zero, for $E \leq E_c$, to a finite value, for $E > E_c$. Note that, for $E > E_c$, the probability converges to a curve which does not depend on the length R . This is consistent with the existence of allowed regions of finite size above the percolation threshold. The number of such regions rapidly decreases when increasing the energy. In a blue-detuned speckle potential [see Fig. 5(a)], there are always forbidden regions but, for energy $E \gtrsim V_R$, almost all percolation regions extend to infinity (P_R quickly approaches the value 1). In a red-detuned speckle potential [see Fig. 5(b)] there is a single percolation region for $E > 0$, as there is no forbidden region in this case.

III. TRANSPORT REGIMES IN A 2D DISORDERED POTENTIAL

In this section, we study the dynamics of a classical particle of fixed total energy E in a 2D anisotropic disorder. The central goal is to characterize the behavior of the mean square displacement as a function of the particle energy. We mainly focus on blue-detuned speckle potentials but also discuss results for the red-detuned case.

A. Classical equations of motion

The dynamics of a classical particle in a disordered potential is most conveniently written by rescaling the dynamical variables in the following way:

$$\tilde{E} = E/|V_R|, \quad (7a)$$

$$\tilde{\mathbf{r}} = \mathbf{r}/\sigma_R, \quad (7b)$$

$$\tilde{\mathbf{p}} = \mathbf{p}/p_R, \quad (7c)$$

$$\tilde{t} = t/\tau_R, \quad (7d)$$

where E is the particle energy, $\mathbf{r} = (x, y)$ its position, $\mathbf{p} = (p_x, p_y)$ its momentum, and t is the time. Introducing the momentum $p_R = \sqrt{m|V_R|}$ and the time $\tau_R = \sqrt{m\sigma_R^2/|V_R|}$, where m is the atomic mass, the classical equations of motion read:

$$\frac{d\tilde{\mathbf{r}}}{d\tilde{t}} = \tilde{\mathbf{p}}, \quad (8a)$$

$$\frac{d\tilde{\mathbf{p}}}{d\tilde{t}} = -\frac{dv(\tilde{\mathbf{r}})}{d\tilde{\mathbf{r}}}, \quad (8b)$$

where $v(\tilde{\mathbf{r}}) = V(\mathbf{r})/|V_R|$ is the reduced disordered potential. Equations (8) highlight the universality class of the classical dynamics: upon the above rescaling by the disorder parameters V_R and σ_R , the classical dynamics only depends on the properties of the reduced potential, $v(\tilde{\mathbf{r}})$, that is on the model of disorder, and in particular on its anisotropy. It is then sufficient to study the particle dynamics for a single set of parameters V_R and σ_R . This is an important difference compared to the quantum dynamics, which also depends on V_R/E_R , where $E_R = \hbar^2/m\sigma_R^2$ [29, 89, 90].

The reduced equations (8), for the speckle potential described in Section II, form the basis of the calculations reported in this paper. For each realization of the disordered potential $V(\mathbf{r})$, we numerically solve Eqs. (8) to calculate the classical phase-space trajectory $(\mathbf{r}(t), \mathbf{p}(t))$, depending on the initial conditions $(\mathbf{r}_0, \mathbf{p}_0)$. We use an adaptive ordinary differential equation algorithm where the time step is adjusted such that the energy is conserved within about 0.5% over the entire evolution. Numerical simulations are run with the fixed initial position $\mathbf{r}_0 = 0$. The initial momentum is chosen with amplitude $|\mathbf{p}_0| = \sqrt{2m[E - V(\mathbf{r}_0)]}$ and, unless specified, along a random direction chosen homogeneously. Disordered speckle potentials are randomly generated and disregarded if $V(\mathbf{r}_0) > E$. They are typically created in a box of linear length $L = 400\sigma_R$ with a grid step $\Delta x = \Delta y = \sigma_R/10$ and repeated periodically over the infinite plane [88]. Quantities of interest (see below) are then obtained by averaging over different realizations of the disordered potential.

B. Time scale of ballistic dynamics

The primary effect of the disordered potential is to modify the ballistic dynamics of particles. A particle of energy E initially moving along the $r \in \{x, y\}$ direction loses the memory of its initial momentum within a characteristic time scale $\tau_r^*(E)$ [91]. In order to identify $\tau_r^*(E)$, we have studied the momentum covariance tensor

$$\text{Cov}_{r,r'}(t) \equiv \langle p_r(t) p_{r'}(0) \rangle. \quad (9)$$

The typical behavior of the diagonal elements of the normalized momentum covariance, $\text{Cov}_r(t) \equiv \text{Cov}_{r,r}(t)/\langle p_r^2(0) \rangle$, is shown in Fig. 6(a) as a function of time and along the $r = x, y$ directions. We have checked

that the off-diagonal elements of Eq. (9) vanish, in our case. The quantity $\text{Cov}_r(t)$ is characterized by a rapid decay and we define $\tau_r^*(E)$ such that $\text{Cov}_r[\tau_r^*(E)] = 0.1$. For an anisotropic speckle disorder, we find $\tau_x^*(E) \neq \tau_y^*(E)$ and $\tau_r^*(E)$ increases with σ_r ($r \in \{x, y\}$). We then introduce $\tau^*(E) \equiv \min[\tau_x^*(E), \tau_y^*(E)]$. For times $t \lesssim \tau^*(E)$, the dynamics is ballistic and isotropic. For times $t \sim \tau^*(E)$, the dynamics is modified by the disorder, it enters various transport regimes and may become anisotropic (see Sec. III D).

Note that, in our simulations, $\text{Cov}_r(t)$ does not exactly converge to zero at large times as exemplified in Fig. 6(a). We indeed find fluctuations with both positive and negative values, which survive massive configuration averaging. We have however checked, from our numerical results, that the integral $I_r(t) \equiv \int_0^t du \text{Cov}_{r,r}(u)/m^2$ converges in the long time limit to a finite value (up to fluctuations), as shown in Fig. 6(b). This holds for particles of energy both above and below the percolation threshold. In addition, for particles in the normal diffusion regime, the limit of this integral equals the diffusion coefficient obtained with the fit of the mean-square displacement as discussed in Sec. III D). This is consistent with the celebrated relation, $D_{r,r'} = \int_0^{+\infty} dt \text{Cov}_{r,r'}(t)/m^2$, valid for normal diffusion [84], where $D_{r,r'}$ is the (r, r') component of the diffusion tensor.

C. Energy redistribution

Another main effect of disorder is to redistribute, after a time of the order of $\tau^*(E)$, the different forms (kinetic and potential) of the particle energy, under the constraint that the total energy is conserved. An example is shown in Fig. 6(c) where we plot the average kinetic energy along the x direction, $\langle p_x(t)^2 \rangle/2m$, as a function of time. Different lines correspond to particles with the same energy $E = V_R$ and different initial momentum directions \mathbf{p}_0 . After times of the order of $\tau^*(E)$, the lines converge to the same value, $\langle E_k \rangle$, which will be calculated below [see Eq. (12)].

A first insight into the dynamics is obtained by using the statistical micro-canonical ensemble, which assumes phase-space ergodicity in the shell of energy E . The probability distribution of the potential energy E_p for a particle of energy E in a single realization of the disordered potential is then $n(E_p) \propto \int d\mathbf{r} d\mathbf{p} \delta[E - \mathbf{p}^2/2m - V(\mathbf{r})] \delta[V(\mathbf{r}) - E_p]$. In a continuous, homogeneous disorder, averaging over realizations of the disordered potential equals spatial averaging, so that $\int d\mathbf{r} \delta[V(\mathbf{r}) - V] \propto P(V)$ [92].

For a blue-detuned speckle potential, using Eq. (2), we find,

$$n(E_p) = \frac{e^{-E_p/V_R}}{V_R(1 - e^{-E/V_R})} \Theta(E_p) \Theta(E - E_p), \quad V_R > 0, \quad (10)$$

normalized such that $\int dE_p n(E_p) = 1$. The distribution $n(E_p)$ is shown in Fig. 6(d) for $E = V_R > 0$. The

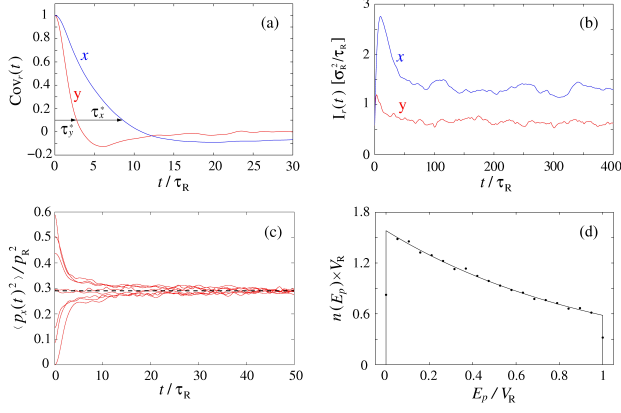


Figure 6: (color online) Primary effects of the disordered potential on the dynamics of a classical particle: modification of the ballistic dynamics and redistribution of energy. Here we consider particles of energy $E = V_R$. Panel (a) shows the normalized momentum covariance $\text{Cov}_r(t)$ for particles with momentum $p_r(0) = \sqrt{2m[E - V(\mathbf{r}_0)]}$ along the $r = x$ (blue line) and $r = y$ (red line) direction, as a function of time. (b) Integral of the momentum covariance, $I_r(t)$, (in units σ_R^2/τ_R), as a function of time and along the $r = x$ (blue line) and $r = y$ (red line) direction. (c) Average kinetic energy, $\langle p_x(t)^2 \rangle / 2m$. The different solid lines refer to particles with fixed energy and different initial momentum directions. The dotted line is Eq. (12). Analogous results are obtained for $\langle p_y(t)^2 \rangle / 2m$. (d) Potential energy distribution obtained at time $t = 30\tau_R$. The dots are numerical results and the solid line is Eq. (10). The numerical results have been obtained for a blue-detuned speckle potential ($V_R > 0$) by averaging over 10000 realizations in panels (a) and (d) and 2000 realizations in panels (b) and (c).

good agreement between the numerical data (dots) and the analytic prediction (Eq. (10); solid line) legitimates the micro-canonical statistical model above, and, consequently, the phase-space ergodicity hypothesis. Using Eq. (10), we find that the average potential energy of a particle of energy E in the disordered potential is

$$\langle E_p \rangle = V_R \left(1 - \frac{E/V_R}{e^{E/V_R} - 1} \right), \quad V_R > 0. \quad (11)$$

Then, using the relation imposed by energy conservation, $E = \langle E_k \rangle + \langle E_p \rangle$, we find that the average kinetic energy is given by

$$\langle E_k \rangle = V_R \left(\frac{E/V_R}{1 - e^{-E/V_R}} - 1 \right), \quad V_R > 0. \quad (12)$$

In particular, in the limit $E \ll V_R$, we have $\langle E_p \rangle \simeq E/2$ and $\langle E_k \rangle \simeq E/2$. This is easily interpreted: The particles are trapped in minima of the disordered potential, which can be assimilated to local (2D) harmonic oscillators, and the total energy is equally partitioned over the four degrees of freedom. In the opposite limit, $E \gg V_R$, the energy of the particles exceeds the disordered potential,

the velocity field is nearly uniform and the average potential energy equals the spatial average of the disordered potential. We have $\langle E_p \rangle \simeq V_R$ and $\langle E_k \rangle \simeq E - V_R$. Finally, Eq. (12) is in excellent agreement with the asymptotic value of the kinetic energy obtained in numerical calculations [see Fig. 6(c)]. For a red-detuned speckle potential, we find a similar behavior. The counter-parts of Eqs. (10), (11) and (12) have been calculated [93] and checked numerically.

D. Energy-dependent transport regimes

In the following we study the mean-square displacement, $\langle r(t)^2 \rangle$, along the $r \in \{x, y\}$ direction. Note that, after disorder averaging, the two parity symmetries $x \leftrightarrow -x$ and $y \leftrightarrow -y$ of the 2D (anisotropic) speckle potential impose $\langle x(t) \rangle = \langle y(t) \rangle = \langle x(t)y(t) \rangle = 0$, for $t \gg \tau^*(E)$ when fixing the initial momentum direction. The above mean values hold at any time when averaging over an isotropic initial momentum distribution.

Figures 1(d)-(f) show the typical behavior of $\langle r(t)^2 \rangle$ as a function of time, for three different energies E in a blue-detuned speckle potential ($V_R > 0$). Quite generally, the plots in log-log scale show straight lines for long time periods. In each of them, the dynamics is well described by the general formula

$$\frac{\langle r(t)^2 \rangle}{\sigma_R^2} = 2\tilde{D}_r \times \left(\frac{t}{\tau_R} \right)^{\alpha_r}, \quad (13)$$

where the parameters \tilde{D}_r and α_r depend on the rescaled energy E/V_R . The different transport regimes are determined by the value of the exponent α_r : (i) classical localization ($\alpha_r = 0$), where $\langle r(t)^2 \rangle$ is independent of time; (ii) sub-diffusion ($0 < \alpha_r < 1$); (iii) normal diffusion ($\alpha_r = 1$); (iv) super-diffusion ($1 < \alpha_r < 2$); (v) ballistic expansion ($\alpha_r = 2$), which holds for free particles.

As discussed above, for times $t \lesssim \tau^*(E)$, the dynamics is isotropic and ballistic. We have

$$\frac{\langle r(t)^2 \rangle}{\sigma_R^2} = \bar{v}_0^2 \times \left(\frac{t}{\tau_R} \right)^2, \quad (14)$$

where $\bar{v}_0 = \sqrt{2\langle E_k \rangle_0 / |V_R|}$ and $\langle E_k \rangle_0$ is the kinetic energy averaged over an isotropic momentum distribution. For the 2D geometry considered here we have $\langle E_k \rangle_0 = \langle E_k \rangle$, which for a blue-detuned speckle potential is given by Eq. (12) [94]. Equation (14) is the dotted black line in Figs. 1(d)-(f) and, as expected, it shows a very good agreement with the numerics [94] for short times.

For longer times ($t \gg \tau^*(E)$), the dynamics is strongly affected by the disorder and we have $\alpha_r < 2$. In the following, we discuss the transport regimes accessible to the particles, depending on their energy.

Classical localization regime. Let us first discuss the dynamical behavior of a particle of energy below the

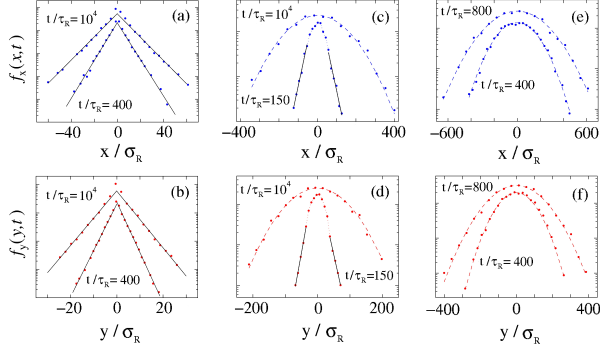


Figure 7: (color online) Integrated, average spatial distribution (dots) along the x (upper row) and y (lower row) directions in semi-log scale. Different sets of data in each figure refer to different evolution times, vertically shifted for clarity and explicitly indicated. Here we consider a blue-detuned speckle potential. In panels (a) and (b) the energy is $E = 0.3V_R$. The solid lines are fit to the function $f_r(r, t) \approx e^{-b_r(t)|r|}$ which reproduces quite well the numerical results in the tails of the distributions. In particular, we have $b_x(t/\tau_R = 10^4) = 0.11/\sigma_R$ and $b_y(t/\tau_R = 10^4) = 0.22/\sigma_R$, the ratio between the two values being equal to the anisotropy factor $\lambda = 2$. In panels (c) and (d) the energy is $E = V_R$. The dotted lines are guides to the eye. The black lines are fits with exponential functions, $f(r, t) \approx e^{-b_r(t)|r|}$, for relatively short times ($t/\tau_R = 150$), while the dashed lines are $f(r, t) \approx e^{-r^2/4D_r(E)t}$, reproducing the data for long times. In panels (e) and (f) the energy is $E = 3V_R$ and the dashed lines are $f(r, t) \approx e^{-r^2/4D_r(E)t}$. The diffusion coefficients $D_r(E)$ are given by Eqs. (15) and (16).

percolation threshold ($E < E_c$). Figure 1(d) shows that, in average, $\langle r(t)^2 \rangle$ is well described by Eq. (13), with different parameters in different time intervals. At intermediate times, the particle chaotically explores the finite-size allowed region it belongs to, bouncing between the borders. Its probability distribution progressively fills this region and we then find a subdiffusive behavior ($\alpha_r < 1$). At asymptotically large times, when the probability distribution has filled all the allowed region, we find $\alpha_r = 0$, corresponding to classical localization. In this case, the coefficients \tilde{D}_r in Eq. (13) are given by the average mean-square size of the allowed regions along the x and y directions [these are plotted as horizontal solid lines in Fig. 1(d)]. In an anisotropic disorder, we find that $\langle x^2 \rangle / \langle y^2 \rangle \simeq \lambda^2$, which is consistent with the fact that the allowed regions are anisotropic, with the same anisotropy factor as the potential.

In order to get further insight on the relation between this behavior and the topography of the allowed regions, we have studied the “probability of diffusion”, $f(\mathbf{r}, t | \mathbf{r}_0, t_0)$, which is the probability distribution to find a particle at position \mathbf{r} at time t assuming that it is at position \mathbf{r}_0 at time t_0 . Due to time translation invariance and space transition invariance (after averaging

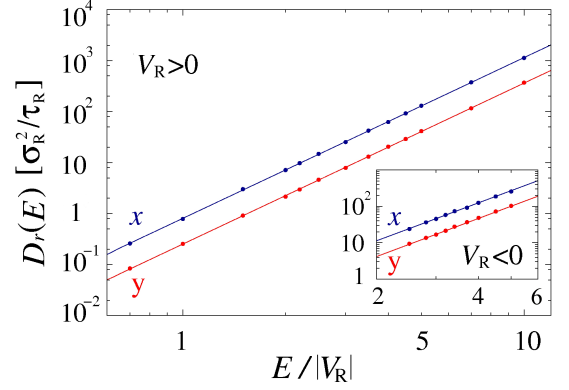


Figure 8: (color online) Diffusion coefficients $D_r(E)$ (dots; in units σ_R^2/τ_R) along the $r = x, y$ directions. The results have been obtained numerically from fits of $\langle r(t)^2 \rangle$, according to Eq. (13), for large times. The main figure shows the results for a blue detuned speckle. Lines are the power-law fit with Eq. (15) and coefficients given by Eq. (16). The inset shows the diffusion coefficients $D_r(E)$ (dots) for a red-detuned speckle disorder, as a function of $E/|V_R| + 2$. Lines are a power-law fit with Eq. (17) and coefficients given by Eq. (18). Numerical results have been obtained after averaging over 2000 realizations of the disordered potential.

ing over disorder configurations), it is sufficient to consider the case $\mathbf{r}_0 = 0$ and $t_0 = 0$. Numerical results for the reduced probabilities $f_x(x, t) = \int dy f(\mathbf{r}, t | \mathbf{r}_0, t_0)$ and $f_y(y, t) = \int dx f(\mathbf{r}, t | \mathbf{r}_0, t_0)$ are shown in Fig. 7(a) and (b) for particles of energy $E = 0.3V_R$ in a blue-detuned speckle potential. The distributions along x and y present tails that can be well fitted by exponential functions. This behavior can be quantitatively related to the probability distribution of the sizes \mathcal{L} of the allowed regions for a given energy E , which is exponentially small, $P_E(\mathcal{L}) \propto \exp(-a_E \mathcal{L}/\sigma_R)$ (see Sec. II B). Consider particles trapped in lakes (allowed region) of linear length \mathcal{L} along x . They chaotically explore the lakes and the linear density along x (averaged over the disorder) is almost uniform and scales as $1/\mathcal{L}$. Hence, the “probability of diffusion” scales as $f_x(x, t \rightarrow \infty) \sim \int_x^\infty d\mathcal{L} P_E(\mathcal{L}) \frac{1}{\mathcal{L}}$. We then find that, for $x \gg \sigma_R/a_E$, $f_x(x, t \rightarrow \infty)$ decays exponentially with the same characteristic length of $P_E(\mathcal{L})$, namely $\ln[f_x(x, t \rightarrow \infty)] \sim -a_E x/\sigma_R$. For $E = 0.3V_R$, we indeed find that, in the long time and long distance limits, the decay rate of $f_x(x, t)$ [$b_x \simeq 0.11/\sigma_R$; see Fig. 7(a)] is approximately the same as that of P_E [$a_E \simeq 0.12/\sigma_R$; see the red triangles in Fig. 4(a)]. The decay of $f_y(y, t)$ [see Fig. 7(b)] is simply related to that of $f_x(x, t)$ by taking into account the anisotropy factor $\lambda = \sigma_x/\sigma_y$ of the disorder. Therefore, the function $f_y(y, t)$ has exponential tails which decay λ times faster than $f_x(x, t)$.

Normal diffusion regime. For sufficiently long times and for energies above the percolation threshold,

the dynamics is diffusive ($\alpha_r = 1$). This is shown in Fig. 1(e) and (f) where the solid lines are fits of $\langle r(t)^2 \rangle = 2D_r(E/V_R) \times t$ to the numerical data for $t \gg \tau^*$, with D_r as fitting parameter (along the $r = x, y$ directions). We have studied the behavior of the diffusion coefficients $D_r(E/V_R)$ as a function of the particle energy, for blue- and red-detuned speckle potentials.

For a blue-detuned speckle potential, we find a clear power-law scaling of the diffusion coefficients versus the energy,

$$D_r(E/V_R) = D_r^0 \times \left(\frac{E}{V_R} \right)^{\gamma_r}, \quad V_R > 0, \quad (15)$$

as evidenced by Fig. 8, which shows D_r versus E in log-log scale. Fits of Eq. (15) to the numerical data, with D_r^0 and γ_r as fitting parameters provide

$$D_x^0 \simeq (0.81 \pm 0.03) \sigma_R^2/\tau_R, \quad V_R > 0, \quad (16a)$$

$$D_y^0 \simeq (0.26 \pm 0.01) \sigma_R^2/\tau_R, \quad (16b)$$

$$\gamma_x \simeq \gamma_y \simeq 3.15 \pm 0.04. \quad (16c)$$

The diffusion coefficients D_r are characterized, within numerical error, by the same exponent γ along the x and y directions. In particular, the anisotropy of the diffusion does not depend much on the energy [95]. Interestingly, we find that, in the diffusion regime, the anisotropy factor of the diffusion, $D_x^0/D_y^0 \simeq 3.1$, significantly differs from the squared anisotropy factor of the disordered potential ($\lambda^2 = 4$). It shows that, differently from the classically localized regime, the anisotropy of the diffusion differs from the anisotropy of the disorder correlation function. This is further investigated in Fig. 9, which shows the ratio $(D_x/D_y)/\lambda^2$ as a function of $\lambda = \sigma_x/\sigma_y$ for $E = 3V_R$. For $\lambda > 1$, we find that $D_x(E)/D_y(E) < \lambda^2$.

For a red-detuned speckle potential, we have

$$D_r(E/V_R) = D_r^0 \times \left(\frac{E - 2V_R}{|V_R|} \right)^{\gamma_r}, \quad V_R < 0. \quad (17)$$

Fits of Eq. (17) to the numerics with D_r^0 and γ_r as fitting parameters are shown in the inset of Fig. 8. We find

$$D_x^0 \simeq (1.19 \pm 0.07) \sigma_R^2/\tau_R, \quad V_R < 0, \quad (18a)$$

$$D_y^0 \simeq (0.37 \pm 0.02) \sigma_R^2/\tau_R, \quad (18b)$$

$$\gamma_x \simeq \gamma_y \simeq 3.45 \pm 0.09. \quad (18c)$$

Compared to the blue-detuned case, we obtain significantly larger coefficients D_r^0 and larger exponents γ_r . However, the anisotropy factor is approximately unchanged, $D_x^0/D_y^0 \simeq 3.2$.

The power law scaling of the diffusion coefficients versus the particle energy in large energy windows is a rather general feature of particle diffusion problems [26, 29, 31]. Here, we find that, remarkably, the exponents $\gamma_{x,y}$ are about the same along the two directions. For the energy window we are considering here, which is relevant to the experiment of Ref. [77], the disorder is strong. This is

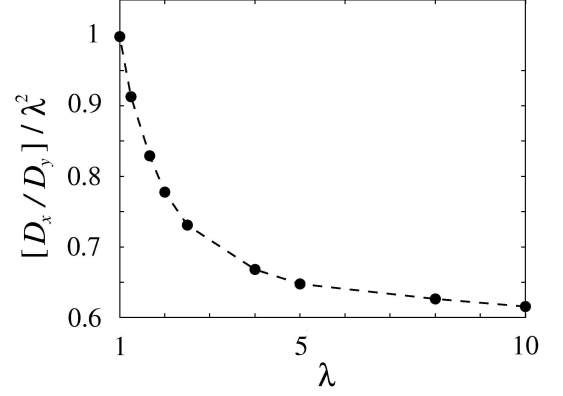


Figure 9: Anisotropy of the diffusion coefficient. Ratio $[D_x/D_y]/\lambda^2$ as a function of $\lambda = \sigma_x/\sigma_y$. Diffusion coefficients are obtained from the fit of $\langle r(t)^2 \rangle$, according to Eq. (13) and for asymptotically large times. Dots are obtained for $E = 3V_R$ and by averaging over 2000 realizations of a blue-detuned speckle potential. The dashed lines is a guide to the eye.

suggested by the fact that the diffusion coefficients at a given energy are significantly different for blue- and red-detuned speckle potentials. For higher energies, the theory of quantum diffusion shows that the diffusion coefficients, calculated in the Born approximation, only depend on the autocorrelation function of the disordered potential [69], which is the same for the blue- and red-detuned speckle potentials used in the present study. For *isotropic* Gaussian correlation functions, one then finds $\gamma = 2.5$ [31, 76]. Note however that the latter result holds for $E \gg V_R^2/E_R$, which can be equivalently written as $(E/V_R)^2 \gg (k\sigma_{x,y})^2$. The parameters used in the present work do not fulfill this condition since the validity of Newton's equation of motion requires $k\sigma_{x,y} \gg 1$ and our calculations are limited to $E \lesssim 10|V_R|$.

In the normal diffusion regime, we expect that the probabilities of diffusion are given by Gaussian functions [67]. In Fig. 7(e) and (f), we plot the (non-normalized) probability distributions along the x and y directions, taken at different times $t = 400\tau_R$ and $t = 800\tau_R$, for a blue-detuned speckle potential and for particles initially at the origin, $\mathbf{r}_0 = 0$. The numerical findings are in very good agreement with the functions $f_r(r, t) \propto e^{-r^2/4D_r(E)t}$, where the coefficients $D_r(E)$ are given by Eqs. (15) and (16). We have checked that this holds for different energies in the normal diffusion regime.

Finally, we have studied the correlation functions $c_k(E, t) = \langle x^k(t)y^k(t) \rangle / \langle x^k(t) \rangle \langle y^k(t) \rangle$ for various integer values of k . For particles in the diffusion regime and for $t \gg \tau^*(E)$, we find $c_k(E, t) \rightarrow 1$ for different values of k . This shows that, in the normal diffusion regime, the x and y variables are independent. Then, taking into account the results of Figs. 7(e) and (f), the probability of diffusion, i.e. the probability density to find a particle at coordinate (x, y) which is in $\mathbf{r}_0 = (x_0, y_0)$ at time t_0 , is

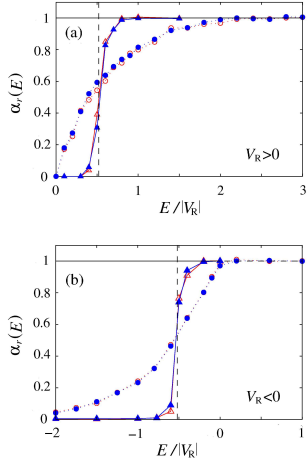


Figure 10: (color online) Scaling coefficient $\alpha_r(E)$ obtained from a fit of $\langle r(t)^2 \rangle$ [according to Eq. (13)] at intermediate times, $100 \lesssim t/\tau_R \lesssim 1000$, (circles) and at larger times, $t/\tau_R \gtrsim 1000$, (triangles). The fit are shown, for instance, in Fig. 1(d)-(f). Solid and dotted lines are guides to the eye. Different colors refer to $r = x$ (blue lines and filled symbols) and $r = y$ (red lines and empty symbols) direction. Panels (a) and (b) refer to blue- and red-detuned speckle potential, respectively. In each panel, the vertical dashed black line is the percolation threshold, $E_c = 0.52V_R$.

given by

$$f_E(\mathbf{r}, t | \mathbf{r}_0, t_0) = \frac{e^{-\frac{(x-x_0)^2}{4D_x(E)(t-t_0)}}}{\sqrt{4\pi D_x(E)}} \frac{e^{-\frac{(y-y_0)^2}{4D_y(E)(t-t_0)}}}{\sqrt{4\pi D_y(E)}} \frac{\Theta(t-t_0)}{(t-t_0)^{1/2}}$$

Sub-diffusion regime. Anomalous diffusion occurs in a variety of physical systems [67, 87, 96] and has been experimentally studied in Refs. [64, 97–99] for instance. It is defined by the condition $\alpha_r \neq 1$ in Eq. (13) and two physically different regimes should be distinguished: sub-diffusion when $0 < \alpha_r < 1$ and super-diffusion when $1 < \alpha_r < 2$. In our case, super-diffusion only shows up for $t \sim \tau^*$ in the crossover from ballistic expansion ($\alpha_r = 2$ for $t \lesssim \tau^*$) to normal or sub-diffusion (for $t \gtrsim \tau^*$). Sub-diffusion is more relevant to our study. For a blue-detuned speckle potential, a transient sub-diffusive behavior is found for $E \lesssim 2V_R$ and long (although not asymptotic) time scales with $t \gg \tau^*$. For instance, the fits of Eq. (13) to $\langle r(t)^2 \rangle$ in Figs. 1(d) and (e), at intermediate times and with \tilde{D}_r and α_r as fitting parameters, (dashed lines) give $\alpha_r < 1$. The scaling coefficients α_r , as obtained from such fits in two different time windows, are plotted as a function of energy in Fig. 10 for blue-detuned (left panel) and red-detuned (right panel) speckle potentials. The value of α_r obtained from the fits in an intermediate time window ($10^2\tau_R < t < 10^3\tau_R$) smoothly crosses over from $\alpha_r = 0$ to $\alpha_r = 1$ when crossing the percolation threshold E_c . For

very long times ($t > 10^3\tau_R$), the crossover is very sharp, which is compatible with an abrupt transition between two asymptotically-relevant dynamical regimes: Below the percolation threshold ($E < E_c$), the mean-square displacement $\langle r(t)^2 \rangle$ evolves from subdiffusion to classical localization [see Fig. 1(e)]. Conversely, slightly above the percolation threshold ($E \gtrsim E_c$), $\langle r(t)^2 \rangle$ evolves from subdiffusion to normal diffusion [see Fig. 1(f)]. Notably, this case shows that the sub-diffusion regime cannot always be considered a precursor of localization.

While the subdiffusive behavior is not asymptotic, it is however experimentally relevant. In fact, subdiffusion is observed for $E \sim V_R$, at times of the order of $\sim 200\tau_R$, which corresponds to $\sim 100\text{ms}$ for the parameters of Ref. [77], that is the typical time scale of the experiment. In the sub-diffusion regime, the probability density $f(\mathbf{r}, t | \mathbf{r}_0, t_0)$ is characterized by exponentially decaying tails [see Figs. 7(a), (b), (c) and (d)]. We interpret this result by analogy with the behavior described above in the classical localization regime. Slightly above the percolation threshold, there exist allowed regions of finite size that are disconnected from each other or connected by very thin bottlenecks to a percolating cluster. Then, as illustrated in Fig. 11(a), a particle stays long times in the same lake where it behaves similarly as below the percolation threshold. In particular, its dynamics is subdiffusive. As discussed above, this persists for asymptotically large times when classical localization occurs [for $E \leq E_c$; Figs. 7(a) and (b)]. Above the percolation threshold ($E > E_c$) however, the particle can find a path connecting the lake to another one after sufficiently long exploration of the lake. It then spends a long time in the new lake before switching to another one, and so on [100]. For longer time scale and larger space scale, the dynamics results from random jumps of the particle from a trapping region to another one, hence developing normal diffusion, similarly as in usual Brownian motion. In particular, the probability of diffusion becomes Gaussian [see Figs. 7(c) and (d) for sufficient long time]. Here, the existence of long waiting times would explain the transient sub-diffusive regime obtained at intermediate times. Note that for higher energies, where no transient sub-diffusion is found, the trajectory of a particle does not show long waiting times and reminds that of Brownian motion even on a short times scale [see Fig. 11(b)].

IV. EXPANSION OF AN ATOMIC CLOUD

Section III presents a detailed analysis of the transport regimes for particles with a fixed energy. In typical experiments with cold atoms, transport is rather investigated from the behavior of a cloud of atoms with a broad energy distribution. In this section, we study the expansion of an atomic gas by considering the atoms as non-interacting classical particles moving in a 2D disordered potential. We take into account the energy distribution of the gas and discuss our findings in the light of the

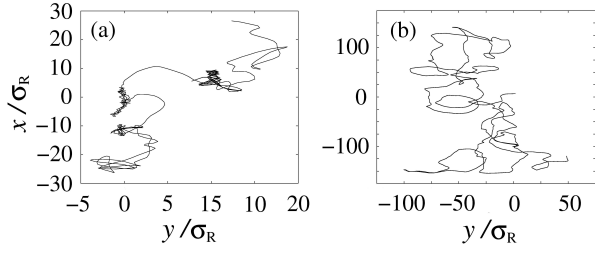


Figure 11: Trajectory of a particle of energy E in an anisotropic, blue-detuned speckle potential, obtained by solving Eqs. (8). In the panel (a) the energy is $E = V_R$ and the trajectory is characterized by relatively small regions where the particle spends long times. This behavior give rise to transient sub-diffusion. The panel (b) shows the trajectory for a particle of energy $E = 3V_R$. The dynamics is diffusive and qualitatively similar to a Brownian motion [67].

results of Sec. III.

A. Energy distribution of an initially trapped gas

The aim of this section is to calculate the energy distribution of a thermal gas of non-interacting particles initially trapped. We will consider two experimentally relevant cases in which the disorder is initially combined with the trap or switched on after releasing the trap. The energy distribution is highly sensitive to the way the disorder is turned on.

Quite generally, in transport experiments with cold atoms, the atomic cloud is initially in equilibrium in a trap described by the potential $V_0(\mathbf{r})$. The initial single-particle classical Hamiltonian is $H_0(\mathbf{r}, \mathbf{p}) = |\mathbf{p}|^2/2m + V_0(\mathbf{r})$ and the phase-space density distribution for a thermal gas at temperature T reads

$$\rho_0(\mathbf{r}, \mathbf{p}) = \frac{e^{-\beta H_0(\mathbf{r}, \mathbf{p})}}{Z_0}, \quad (20)$$

where $Z_0 = \int d\mathbf{r}' d\mathbf{p}' e^{-\beta H_0(\mathbf{r}', \mathbf{p}')}$, $\beta = 1/k_B T$ and k_B is the Boltzmann constant. At time $t = 0$, the confinement is switched off and the non-interacting gas expands in the disordered potential $V(\mathbf{r})$. The dynamics is then governed by the single-particle Hamiltonian $H(\mathbf{r}, \mathbf{p}) = |\mathbf{p}|^2/2m + V(\mathbf{r})$ and the joint distribution of energy and position at time $t = 0^+$ is

$$\begin{aligned} n(\mathbf{r}, E) &= \int d\mathbf{p} \rho_0(\mathbf{r}, \mathbf{p}) \delta(E - H(\mathbf{r}, \mathbf{p})) \\ &= \frac{\beta e^{-\beta(E - V(\mathbf{r}) + V_0(\mathbf{r}))}}{\int d\mathbf{r}' e^{-\beta V_0(\mathbf{r}')}} \Theta(E - V(\mathbf{r})). \end{aligned} \quad (21)$$

In the following, we calculate the disorder-averaged energy distribution $\langle n(\mathbf{r}, E) \rangle$, distinguishing the two experimentally relevant situations: (i) The gas is prepared in a bare trap, so that $V_0(\mathbf{r}) = V_{\text{trap}}(\mathbf{r})$, where $V_{\text{trap}}(\mathbf{r})$ is the

potential of the trap, which is independent of the disordered potential. The disorder is suddenly switched on after the gas has thermalized in the trap. (ii) The gas is prepared in the trap and in the presence of disorder, so that $V_0(\mathbf{r}) = V_{\text{trap}}(\mathbf{r}) + V(\mathbf{r})$, which now depends on the disordered potential.

Case (i). If $V_0(\mathbf{r})$ does not depend on the disordered potential, averaging the distribution Eq. (22) gives

$$\langle n(\mathbf{r}, E) \rangle = \frac{\beta e^{-\beta V_0(\mathbf{r})}}{\int d\mathbf{r}' e^{-\beta V_0(\mathbf{r}')}} \int_{-\infty}^E dV P(V) e^{-\beta(E-V)},$$

where $P(V)$ is the one-point probability distribution of the disordered potential. It is interesting to notice that, in a homogeneous disorder and independently of the specific form of the initial trapping potential, the energy and position variables decouple:

$$\langle n(\mathbf{r}, E) \rangle = n_{\text{trap}}(\mathbf{r}) \langle n(E) \rangle, \quad (23)$$

where

$$n_{\text{trap}}(\mathbf{r}) = \int d\mathbf{p} \rho_0(\mathbf{r}, \mathbf{p}) = \frac{e^{-\beta V_0(\mathbf{r})}}{\int d\mathbf{r}' e^{-\beta V_0(\mathbf{r}')}} \quad (24)$$

is the initial spatial distribution and

$$\langle n(E) \rangle = \beta \int_{-\infty}^E dV P(V) e^{-\beta(E-V)} \quad (25)$$

is the average energy distribution in the presence of disorder. For a speckle potential, Eq. (2) holds and the integral in Eq. (25) can be easily done. In particular, for a blue-detuned speckle potential, we find [104]

$$\langle n(E) \rangle = \frac{\beta}{1 - \beta V_R} \left[e^{-\beta E} - e^{-E/V_R} \right] \Theta(E). \quad (26)$$

A plot of Eq. (26) as a function of the energy is shown in Fig. 12(a) for different values of βV_R .

Case (ii). If on the contrary, $V_0(\mathbf{r})$ depends on the disordered potential, the calculation of Eq. (22) is difficult since both the numerator and the denominator depend on the specific disorder configuration. A simplified expression is obtained if the thermal gas extends, in the initial trapping potential $V_0(\mathbf{r})$, over a region much larger than the correlation length of the disorder. This condition can be fulfilled for a sufficiently shallow confining trap. In this case, the self-averaging properties of the disordered potential allow us to substitute the integral $\int d\mathbf{r}' e^{-\beta V_0(\mathbf{r}')}$ by the disorder-averaged value $\langle \int d\mathbf{r}' e^{-\beta V_0(\mathbf{r}')} \rangle = \int d\mathbf{r}' e^{-\beta V_T(\mathbf{r}')} \int dV P(V) e^{-\beta V}$. We thus find that, also in this case, the energy and position variables decouple. Then, Eq. (23) still holds, with the same density distribution, given by Eq. (24), but a different energy distribution,

$$\langle n(E) \rangle = \frac{\beta e^{-\beta E} \int_{-\infty}^E dV P(V)}{\int_{-\infty}^{+\infty} dV P(V) e^{-\beta V}}. \quad (27)$$

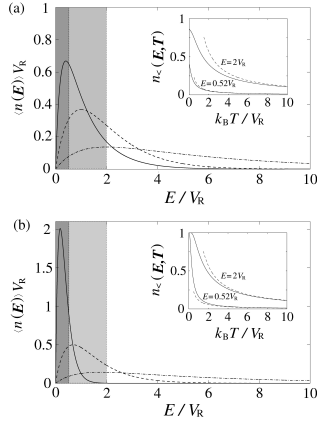


Figure 12: Disorder-averaged energy distributions of the atomic cloud. Panel (a) is Eq. (26), obtained when the speckle potential is suddenly switch on after releasing the trap where the atomic cloud is initially in equilibrium [Case (i)]. Panel (b) is Eq. (28) and corresponds to a thermal cloud in equilibrium in a disordered trap [Case (ii)]. The different curves refer to different values of V_R : $V_R = 5 k_B T$ (solid line), $V_R = k_B T$ (dashed line) and $V_R = 0.2 k_B T$ (dot-dashed line). The gray shaded region indicates sub-diffusive particles ($E \leq 2 V_R$), the darkest region classically localized particles ($E \leq 0.52 V_R$) [see Sec. III]. Inset: density of particles with energy smaller than E (solid lines). The dashed lines are a prediction in the large temperature limit $k_B T \gg V_R$, see footnote [105].

In particular, for a blue-detuned speckle potential, we have [104]

$$\langle n(E) \rangle = \beta(1 + \beta V_R) e^{-\beta E} \left[1 - e^{-E/V_R} \right] \Theta(E). \quad (28)$$

In the case of an isotropic harmonic trap $V_{\text{trap}}(\mathbf{r}) = m\omega^2|\mathbf{r}|^2/2$, we find $n_{\text{trap}}(\mathbf{r}) = e^{-|\mathbf{r}|^2/2\sigma_T^2}/2\pi\sigma_T^2$, where $\sigma_T = \sqrt{k_B T/m\omega^2}$. The self-averaging condition invoked above is thus $\sigma_T \gg \sigma_R$. A plot of Eq. (28) as a function of energy is shown in Fig. 12(b) for different values of βV_R . Equations (26) and (28) tend, in the high-temperature limit, $k_B T \gg E, V_R$, to $\langle n(E) \rangle = \beta(1 - e^{-E/V_R})$. At low temperature, $k_B T \ll V_R$, the two distributions differ substantially, Eq. (28) being more strongly peaked at small energies.

The insets of Fig. 12 show the fraction of particles with energy lower than E , $n_<(E, T) \equiv \int_0^E d\epsilon \langle n(\epsilon) \rangle$ [105], as a function of temperature and for various energies. In particular it shows the fraction of atoms in the classically localized regime, $n_<(E = 0.52 V_R, T)$, and below the transient sub-diffusive regime threshold, $n_<(E = 2 V_R, T)$, identified in Sec. III. The dotted line is the asymptotic behavior for large temperatures, $n_<(E, T) = (e^{-E/V_R} + E/V_R - 1)\beta V_R$, which is the same for both cases (i) and (ii). When the gas thermalizes in the trap and in the presence of disorder [case (ii) described by Eq. (28)], the fraction of particles at low energy is larger than in the

case when the disordered is suddenly ramped after releasing the confining trap [case (i) described by Eq. (26)]. In case (ii), all particles can be classically localized at low temperature [i.e. $n_<(E = 0.52 V_R, T) \rightarrow 1$ when $T \rightarrow 0$]. Conversely, in case (i), the speckle potential is switched on after releasing the initial confining trap, which transfers an average energy $\sim V_R$ to all particles and the fraction of particles at low energy is depleted.

B. Expansion of the atomic gas

We now study the dynamics of an atomic cloud initially trapped in a pure harmonic potential [i.e. without disorder, case (i)]. At $t = 0$ the trap is released and the atoms expand in a blue-detuned speckle potential. As shown above, the energy and the initial position of the atoms are uncorrelated variables, when averaging over disorder configurations. In our simulations, the energy is randomly chosen according to Eq. (26), while the initial position is randomly chosen according to a Gaussian distribution of width $\sigma_T = 10\sigma_R$, which is relevant to typical experiments [77]. We expect this approach to be reproducible *in a single run of the experiment* since, for $\sigma_T \gg \sigma_R$, the speckle disorder “self-averages” over the atomic cloud.

Density distribution. An important feature of the experiments with cold atoms is the possibility to obtain an image of the density cloud at different expansion times. For instance, it has been crucial in the recent observation of Anderson localization with Bose-Einstein condensates in the quasi-1D geometry [61, 62].

The time-dependent density profile (averaged over different realizations of the disordered potential) is calculated by integrating the probability of diffusion over all energies and all initial positions

$$n(\mathbf{r}, t) = \int d\mathbf{r}_0 \int dE n_{\text{trap}}(\mathbf{r}_0) \langle n(E) \rangle f_E(\mathbf{r}, t | \mathbf{r}_0, 0), \quad (29)$$

where $n_{\text{trap}}(\mathbf{r}_0)$ is the spatial density distribution in the initial trap. As noticed in the previous paragraph, for a thermal gas, the initial joint position-energy distribution decouples into the product of separate distributions. This feature is particular to the case of an initial thermal distribution. For instance, it does not hold for initially expanding Bose-Einstein condensates as considered in Refs. [31, 35, 75]. The quantity $f_E(\mathbf{r}, t | \mathbf{r}_0, 0)$ gives the probability distribution to find a particle of energy E at position \mathbf{r} and at time $t \geq 0$, assuming it was in \mathbf{r}_0 at time $t = 0$. Its behavior has been discussed in Sec. III for particles of fixed energy in various transport regimes. The integrated density profiles at time $t = 400\tau_R$ are shown as thick solid lines in Fig. 13. It is striking that these profiles are in general, not Gaussian. They present a pronounced central peak, which is partially due to the contributions of the atoms in the classical localization and sub-diffusive

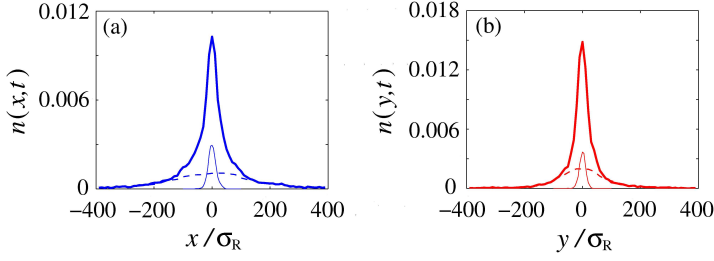


Figure 13: (color online) Integrated spatial density distribution of an expanding thermal gas in a 2D disordered potential. The thick solid line is the integrated spatial density distribution (a) $n_x(x, t) \equiv \int dy n(\mathbf{r}, t)$ and (b) $n_y(y, t) \equiv \int dx n(\mathbf{r}, t)$, where $n(\mathbf{r}, t)$ is given by Eq. (29). Here, $V_R = k_B T$, $t = 400\tau_R$, the initial distribution is a Gaussian of width $\sigma_T = 10\sigma_R$ and the energy distribution is given by Eq. (26). In the numerical simulation, we used 10000 particles. We also plot the contributions of particles with energy $E > 2V_R$ (dashed line) and with energy $E \leq 0.52V_R$ (thin solid line). Note that for the time considered here, the transient sub-diffusive regime is relevant and the fraction of atoms in this regime cannot be neglected.

regimes. It is worth stressing that, even taking into account only the fraction of atoms in the diffusive regime, for which the probability of diffusion, $f_E(\mathbf{r}, t|\mathbf{r}_0, 0)$, is a Gaussian function [see Eq. (19)], the overall distribution is, in general, not Gaussian. This is due to the integration over the energy variable in Eq. (29) and to the strong energy dependence of the diffusion coefficients [75, 76]. In fact, since the particles in the diffusive regime determine the long tails of $n(\mathbf{r}, t)$, the spatial distribution can be used to extract the diffusion coefficients from experimental data (see Ref. [77] and Sec. V).

Mean square size. A simple quantity which can be extracted from the density profiles is the rms size of the atomic cloud, $\Delta r(t) \equiv \sqrt{\int d\mathbf{r} r^2 n(\mathbf{r}, t)}$, in the $r = x, y$ direction and where $n(\mathbf{r}, t)$ is given by Eq. (26). The evolution of the rms sizes in the two directions are shown in Fig. 14, for $V_R = k_B T$. At sufficiently short times, $t \ll \tau^*(\bar{E})$, where \bar{E} is the typical energy of the atomic cloud [107], the rms size can be obtained by assuming that all the atoms behave ballistically [106]. In this case, we find

$$[\Delta r(t)]^2 = \sigma_T^2 + \frac{1 + \beta V_R}{\beta V_R} \sigma_R^2 \left(\frac{t}{\tau_R} \right)^2, \quad (30)$$

where $\sigma_T = \sqrt{k_B T / m \omega^2}$ is the rms size of the initial atomic cloud. Equation (30) is shown by the dotted black line in Fig. 14. For long times, $t \gg \tau^*(\bar{E})$, the rms size of the atomic cloud can be calculated by assuming that all the particles behave diffusively. This approximation is legitimate in the long-time limit if the fraction of classically localized particles is negligible, which strongly depends on the ratio $k_B T / V_R$. In this case, the evaluation

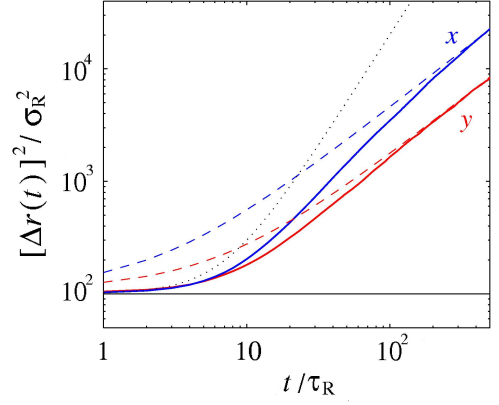


Figure 14: (color online) Mean square size, $[\Delta r(t)]^2$, as a function of time, in the $r = x$ (solid blue line) and $r = y$ (solid red line) directions. The dotted line is Eq. (30) and has been obtained by assuming a ballistic expansion for all atoms. The dashed lines are Eq. (31), corresponding to all particles behaving diffusively. The horizontal line is the mean square size of the initial cloud, given by σ_T^2 . Here $V_R = k_B T$ and $\sigma_T = 10\sigma_R$.

of Eq. (29) using Eqs. (26), (19) and (15) gives

$$[\Delta r(t)]^2 \simeq 2D_r^0 \sigma_R^2 \frac{[1 - (\beta V_R)^{\gamma_r + 1}] \Gamma(\gamma_r + 1)}{(1 - \beta V_R)(\beta V_R)^{\gamma_r}} \frac{t}{\tau_R}, \quad (31)$$

where $\Gamma(x)$ is the Gamma function. In Fig. 14 we show Eq. (31) calculated for the parameters D_r^0 and γ_r given by Eqs. (16). We find a good agreement with the numerical findings for long times, despite the fact that the fraction of localized or sub-diffusive atoms is $n_{<}(E = 2V_R, T) \simeq 70\%$. Hence, Eq. (31) could be used experimentally to extract the values of γ_r and D_r^0 characterizing the diffusive regime, and thus verify the results of Sec. III D.

V. EXPERIMENTAL RESULTS

The study of classical transport of cold gases in a 2D disordered potential is experimentally feasible. A 2D disordered potential can be created experimentally by a speckle field, as discussed in Sec. II A. The atoms can be confined to a plane by a tight transversal harmonic trap of frequency ω_z . The dynamics in the transverse direction can be neglected if $k_B T$ is much smaller than the vertical confinement energy $\hbar \omega_z$.

In recent experiments [77], we have studied the dynamics of a cloud of ^{87}Rb atoms in a speckle disorder. The geometry is not strictly 2D since $k_B T \approx k_B \times 200(20) \text{ nK} \approx 6\hbar \omega_z$, where $\omega_z/2\pi = 680 \text{ Hz}$. Therefore, taking into account the harmonic confinement, the external potential is given by $V(x, y, z) + m\omega^2 z^2/2$, with $V(x, y, z) = V_{\text{iso}}(x \sin \theta - z \cos \theta, y)$ where $V_{\text{iso}}(u, v)$ is a

2D isotropic blue-detuned speckle potential with correlation length $\sigma = 0.8 \mu\text{m}$ and Gaussian autocorrelation function. In the experiment, the speckle disorder is tilted by $\theta = 30^\circ$ with respect to the horizontal plane. This makes the disorder anisotropic in the expansion plane, with correlation lengths $\sigma_x = 1.6 \mu\text{m}$ and $\sigma_y = 0.8 \mu\text{m}$, and anisotropy factor $\lambda = \sigma_x/\sigma_y = 2$. The speckle is assumed to be invariant along its propagation axis. This is a reasonable approximation since the longitudinal correlation length, $\sigma_{\text{long}} = 9 \mu\text{m}$ is large compared to the vertical size of the cloud, $(k_B T/m\omega_z^2)^{1/2} \approx 1 \mu\text{m}$.

Using a classical particle model is a reasonable approximation since $k\sigma_x \approx 10$ and $k\sigma_y \approx 5$, where $k = \sqrt{mk_B T}/\hbar$. In addition, the ‘‘correlation energy’’ $E_R = \hbar^2/m\sigma_R \approx k_B \times 2 \text{ nK}$ is typically more than one order of magnitude smaller than the amplitude of the disorder $V_R = k_B \times 53 \text{ nK}$. Within these conditions, the localization length should be much larger than the system size ($\sim 1 \text{ mm}$ in each direction) and one can expect that quantum interference effects can be neglected. Due to the tilting angle of the speckle pattern, which couples the in-plane dynamics to the dynamics in the confined direction, the system is 3D rather than purely 2D. Therefore, the classical equations of motions (8) must be adapted to take into account the vertical confinement [108].

Remarkably, the results of this 3D model differ only slightly from the results presented above for the purely 2D situation. The two asymptotic regimes (classical localization and normal diffusion) are unchanged since the 3D potential results from a cylindrical stretching of a 2D potential. We also find a transient sub-diffusion regime in approximately the same energy window and time scale as for the pure 2D case, $0.52V_R \lesssim E \lesssim 2V_R$. For each regime, the dynamics is qualitatively similar for both the pure 2D and the trapped 3D cases. The diffusion coefficients have a power law scaling analogous to Eq. (15) [77]. The numerical simulations give diffusion parameters slightly different to the ones of Eq. (16):

$$D_x^0 = (1.20 \pm 0.05) \sigma_R^2/\tau_R, \quad (32a)$$

$$D_y^0 = (0.33 \pm 0.02) \sigma_R^2/\tau_R, \quad (32b)$$

$$\gamma_x \simeq \gamma_y = 2.8 \pm 0.06. \quad (32c)$$

For the parameters of the experiment, $V_R = k_B \times 53 \text{ nK}$, $\sigma_R = \sigma_x = 1.6 \mu\text{m}$ and $\tau_R = 0.71 \text{ ms}$, we obtain $D_x^0 = 4.3 \mu\text{m}^2/\text{ms}$ and $D_y^0 = 1.2 \mu\text{m}^2/\text{ms}$.

In the experiment, the speckle disorder is switched on after releasing the initial confining trap in the (x,y) direction, while the transverse confinement is kept during the whole expansion. We expect that a few collisions redistribute the energy between the atoms during the first 10 ms of the expansion. The calculation of $\langle n(E) \rangle$ is modified compared to Eq. (28) in order to include the contribution of the different vertical energy levels. Taking into account the quantization of the transverse harmonic

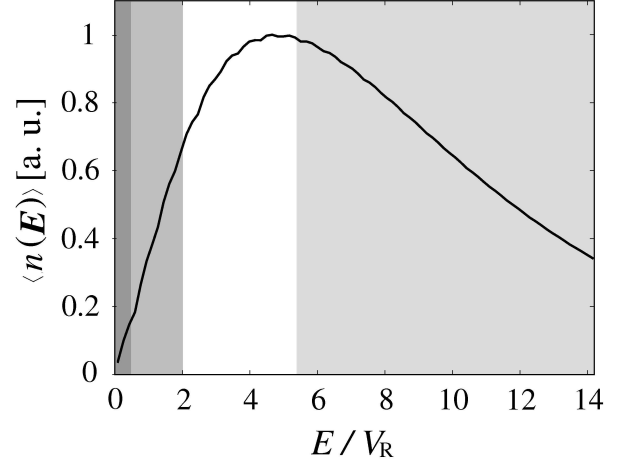


Figure 15: Disorder-averaged energy distribution in the experiments of Ref. [77]. The solid line is Eq. (33). Gray regions indicate different dynamical regimes: classical localization for $E < 0.52V_R$ (darkest region), transient subdiffusion for $0.52V_R < E < 2V_R$ (gray region), diffusion for $2V_R < E < 5.5V_R$ (white region) and transient ballistic expansion (see text for details) for $E > 5.5V_R$ (light gray region). Here the amplitude of the disordered potential is $V_R = k_B \times 53 \text{ nK}$ and the temperature $T = 220 \text{ nK}$.

oscillator, we obtain

$$\langle n(E) \rangle \propto e^{-E/k_B T} \sum_{n=0}^{E/\hbar\omega_z} \left(1 - e^{(n\hbar\omega_z - E)/V_R} \right) \quad (33)$$

which is plotted in Fig. 15. For the considered experimental parameters, the fraction of classically localized atoms is about 0.4% of the total number of atoms while the fraction of subdiffusive atoms for the time scale of the experiment is about 6%.

The goal of the experiment is to extract the energy-dependent diffusion coefficients in the 2D anisotropic speckle potential. They are experimentally measured by analyzing the complete 2D column density profile, obtained experimentally by fluorescence imaging along the vertical axis (see Fig. 16). In order to fit the experimental density, we use Eqs. (19) (with $t_0 = 0$) and (29) convolved by a Gaussian function which takes into account the resolution of the imaging system. Using the fact that, for an initial harmonic trap, $n_{\text{trap}}(\mathbf{r}_0)$ is a Gaussian function, we find

$$n(x, y, t) = \int dE \langle n(E) \rangle \frac{e^{-\frac{x^2}{2(2D_x(E)t + \sigma_0^2)}} e^{-\frac{y^2}{2(2D_y(E)t + \sigma_0^2)}}}{2\pi t \sqrt{(2D_x(E)t + \sigma_0^2)(2D_y(E)t + \sigma_0^2)}}, \quad (34)$$

where $\langle n(E) \rangle$ is given by Eq. (33) and $\sigma_0 = 15 \mu\text{m}$ as determined experimentally. Equation (34) assumes that all the particles behave diffusively. This is a good approximation since, for the experimental parameters, the fraction of classically localized and subdiffusive particles is small. Diffusion coefficients are extracted by using

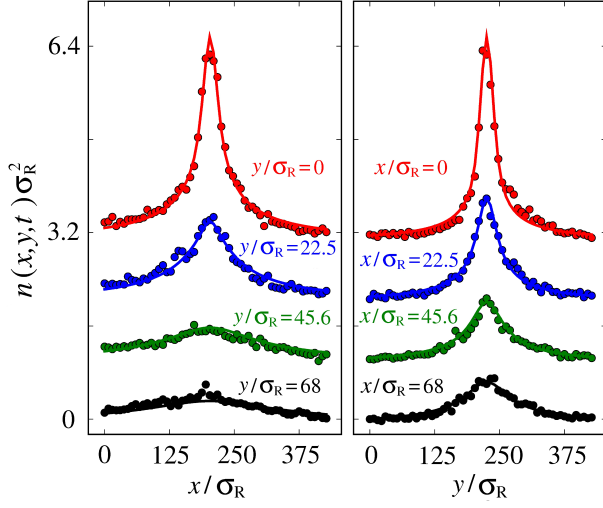


Figure 16: (color online) Experimental two-dimensional column density distribution after 200 ms ($= 281\tau_R$) expansion of the atomic cloud. Dots are cuts along x (left panel) and y (right panel) directions at cut positions explicitly indicated, artificially offset for clarity. Lines are the fitting function Eq. (34). Note that the spatial distribution is not Gaussian.

the power law scaling predicted numerically, $D_r(E) = D_r^0 \times (E/V_R)^{\gamma_r}$ ($r = x, y$). Figure 16 shows that the 2D density distribution is well reproduced by the fitting function, Eq. (34), with D_r^0 and γ_r as fitting parameters. Experimentally, we find:

$$D_x^0 = 5.5 \pm 2.7 \mu\text{m}^2/\text{ms}, \quad (35a)$$

$$D_y^0 = 1.6 \pm 0.8 \mu\text{m}^2/\text{ms}, \quad (35b)$$

$$\gamma_x \simeq \gamma_y = 3.3. \quad (35c)$$

These values are, within experimental errors, in fair agreement with the classical theory [see Eqs. (32)]. In particular, the algebraic scaling inferred from the numerical simulations is consistent with the experimental findings. In addition, these results show the possibility to extract the single-particle diffusion coefficients from the dynamics of an atomic cloud.

We finally notice that the high energy atoms are ballistic on the time scale of the experiment (see Fig. 15). The crossover between the ballistic and the diffusive behavior takes place when the Boltzmann time $\tau_B (= 4D(E)/v^2$, where $v = \sqrt{E/m}$ is the rms in-plane velocity) is of the order of the expansion time $t = 200$ ms. The same crossover between ballistic and diffusive behaviors is seen as a function of time in Fig. 1(c) for a given energy. For the experimental parameters, we estimate that below $E_{\text{bal-dif}} \approx 5.5V_R$, the atoms are diffusive (calculated in the x direction which gives the lower estimate of $E_{\text{bal-dif}}$). Above this value, the atoms may be ballistic on the time scale of the experiment. In this case they however extend over a distance $vt \gtrsim 1$ mm, larger than the spatial size used in Fig. 16 where we fit our diffusive distribution. Ballistic atoms thus do not affect the result of the fit for

the diffusion coefficients. Note that this is not the case after 50 ms as ballistic atoms are visible (see Fig. 2 in Ref. [77]).

In addition to the investigation of the normal diffusion regime discussed above [77], we have recently performed experiments with a lower temperature of 30 nK. In this case, Eq. (33) leads to 8% of classically trapped atoms and 55% of atoms in the subdiffusive regime. We observe little expansion of the cloud as a function of time. For times of several seconds we observe losses due to residual heating. We are unable to quantitatively analyze the shape of the cloud because of the limited resolution of the imaging system and of the initial size of the cloud. The subdiffusive regime and the percolation transition presented in this manuscript thus remain to be experimentally studied in 2D cold atomic systems.

VI. CONCLUSIONS

In this work, we have studied the transport properties of a cold atomic gas in a 2D anisotropic speckle potential, in the classical regime. We have shown that, upon proper rescaling, the atomic dynamics can be parametrized by the energy as the sole relevant quantity. Then, we have identified two asymptotically-relevant regimes, depending on the particle energy: (i) For energy below the percolation threshold, any particle is trapped in a finite-size region, so that $\langle r^2 \rangle \rightarrow \text{const}$ (classical localization regime), along the $r = x, y$ direction. The probability of diffusion (defined as the average density probability of finding an atom at a given distance from its origin point) develops exponentially decaying tails, which correspond to the probability distribution of the size of the classically-localized regions. Due to the pure topographical nature of the latter, the probability of diffusion is anisotropic with the same anisotropy factor as the speckle potential. (ii) For energy above the percolation threshold, there appear infinite-size allowed regions in which the gas expands diffusively, $\langle r^2 \rangle \propto t$ (normal diffusion regime). The probability of diffusion is a Gaussian function with anisotropy factor (along the x and y directions) significantly different from the anisotropy factor of the speckle potential. The diffusion coefficients along the two main directions of the speckle potential strongly depend on the particle energy. More precisely, we found a power-law scaling, $D_r \sim E^\gamma$ with $\gamma \simeq 3.15$ for a blue-detuned speckle potential and $\gamma \simeq 3.45$ for a red-detuned speckle potential.

In addition, a transient sub-diffusion regime, where $\langle r^2 \rangle \propto t^{\alpha_r}$ with $\alpha_r < 1$, characterizes intermediate time scales, relevant to recent experiments [77]. This regime appears below and slightly above the percolation threshold. It is interpreted as due to the dynamics of exploration of finite-size allowed regions which are disconnected from each other or connected by narrow bottlenecks. For energy below the percolation threshold, the long-time dynamics crosses over to classical localization

($\alpha_r = 0$). Conversely, above the percolation threshold, it crosses over to normal diffusion ($\alpha_r = 1$). This example shows that subdiffusion should not be regarded as precursor of classical localization.

Using the above results as a groundwork, we have studied the expansion dynamics of a cloud of cold atoms with broad energy distributions, as relevant to recent experiments. First, we have calculated the energy distribution of the cloud in the presence of disorder. Then, we have discussed the dynamics. The time-dependent profiles are in general anisotropic and non-Gaussian, with tails governed by diffusive atoms. We have proposed a method to extract the energy-dependent diffusion coefficients from these tails, which involve many energy components. The experiments of Ref. [77] are realized in a quasi-2D geometry, which does not significantly change the above-discussed dynamics. The energy-dependent diffusion coefficients extracted from experimental data are in fair agreement with numerical calculations performed in the precise experimental geometry.

In the future, it would be interesting to experimentally study the classical localization regime as discussed in the paper. In particular, working at lower temperature, it would be possible to create a gas of atoms with a significant fraction below the percolation threshold. Then, studying the fraction of classically localized atoms versus temperature, for instance, would give direct access to the percolation transition and, in particular, to the crit-

ical exponent. Another interesting direction would be to extend the study of transport in anisotropic 2D speckle potentials in the quantum regime, where traces of weak and strong localization effects can be searched for.

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References

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- [1] P.A. Lee and T.V. Ramakrishnan, *Rev. Mod. Phys.* **57**, 287 (1985).
 - [2] J.D. Reppy, *J. Low Temp. Phys.* **87**, 205 (1992).
 - [3] E. Akkermans and G. Montambaux, *Mesoscopic Physics of Electrons and Photons* (Cambridge University Press, 2007).
 - [4] S. John, *Phys. Today* **44**, 32 (1991).
 - [5] S. Chandrasekhar, *Radiative Transfer* (Dover, New York, 1960).
 - [6] N.W. Ashcroft and N.D. Mermin, *Solid State Physics* (Saunders, Philadelphia, USA, 1976).
 - [7] D. Vollhardt and P. Wölfle, *Phys. Rev. B* **22**, 4666 (1980).
 - [8] P.W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
 - [9] P.-G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).
 - [10] N.F. Mott, *Rev. Mod. Phys.* **40**, 677 (1968).
 - [11] N.F. Mott, *Metal-Insulator Transitions* (Taylor and Francis, London, 1990).
 - [12] D. Vollhardt and P. Wölfle, in *Electronic phase transitions*, edited by W. Hanke and Y.V. Kopayev (Elsevier, Amsterdam, 1992).
 - [13] L. Fallani, C. Fort, and M. Inguscio, *Adv. At. Mol. Opt. Phys.* **56**, 119 (2008).
 - [14] L. Sanchez-Palencia and M. Lewenstein, *Nature Phys.* **6**, 87 (2010).
 - [15] G. Modugno, *Rep. Prog. Phys.* **73**, 102401 (2010).
 - [16] A. Lagendijk, B.A. van Tiggelen, and D.S. Wiersma, *Phys. Today* **62**, 24 (2009).
 - [17] A. Aspect, and M. Inguscio, *Phys. Today* **62**, 30 (2009).
 - [18] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
 - [19] S. Giorgini, L.P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **80**, 1215 (2008).
 - [20] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen (De), and U. Sen, *Adv. Phys.* **56**, 243 (2007).
 - [21] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
 - [22] D. Clément, A.F. Varón, J.A. Retter, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, *New. J. Phys.* **8**, 165 (2006).
 - [23] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, and M. Lewenstein, *Phys. Rev. Lett.* **91**, 080403 (2003).
 - [24] T. Roth and K. Burnett, *J. Opt. B: Quant. Semiclass. Opt.* **5**, S50 (2003).
 - [25] U. Gavish and Y. Castin, *Phys. Rev. Lett.* **95**, 020401 (2005).
 - [26] R.C. Kuhn, C. Miniatura, D. Delande, O. Sigwarth, and C.A. Müller, *Phys. Rev. Lett.* **95**, 250403 (2005).
 - [27] P. Massignan and Y. Castin, *Phys. Rev. A* **74**, 013616 (2006).
 - [28] L. Sanchez-Palencia, D. Clément, P. Lugan, P. Bouyer, G.V. Shlyapnikov, and A. Aspect, *Phys. Rev. Lett.* **98**, 210401 (2007).
 - [29] R.C. Kuhn, O. Sigwarth, C. Miniatura, D. Delande, and C.A. Müller, *New. J. Phys.* **9**, 161 (2007).
 - [30] S.E. Skipetrov, A. Minguzzi, B.A. van Tiggelen, and B. Shapiro, *Phys. Rev. Lett.* **100**, 165301 (2008).

- [31] C. Miniatura, R.C. Khun, D. Delande, and C.A. Müller, *Eur. Phys. J. B* **68**, 353 (2009).
- [32] L. Pezzé, B. Hambrecht and L. Sanchez-Palencia, *Europhys. Lett.* **88**, 30009 (2009).
- [33] L. Pezzé and L. Sanchez-Palencia, *Phys. Rev. Lett.* **106**, 040601 (2011).
- [34] M. Antezza, Y. Castin, and D. Hutchinson, *Phys. Rev. A* **82**, 043602 (2010).
- [35] M. Piraud, P. Lugan, P. Bouyer, A. Aspect and L. Sanchez-Palencia, *Phys. Rev. A*, **83**, 031603 (2011).
- [36] N. Bilas and N. Pavloff, *Eur. Phys. J. D* **40**, 387 (2006).
- [37] P. Lugan, D. Clément, P. Bouyer, A. Aspect, M. Lewenstein, and L. Sanchez-Palencia, *Phys. Rev. Lett.* **98**, 170403 (2007).
- [38] P. Lugan, D. Clément, P. Bouyer, A. Aspect, and L. Sanchez-Palencia, *Phys. Rev. Lett.* **99**, 180402 (2007).
- [39] P. Lugan and L. Sanchez-Palencia, *Phys. Rev. A*, in press (2011) e-print: arXiv:1105.0610.
- [40] T. Paul, P. Schlagheck, P. Leboeuf, and N. Pavloff, *Phys. Rev. Lett.* **98**, 210602 (2007).
- [41] B. Horstmann, J.I. Cirac, and T. Roscilde, *Phys. Rev. A* **76**, 043625 (2007).
- [42] G. Roux, T. Barthel, I.P. McCulloch, C. Kollath, U. Schollwoeck, and T. Giamarchi, *Phys. Rev. A* **78**, 023628 (2008).
- [43] T. Paul, M. Albert, P. Schlagheck, P. Leboeuf, and N. Pavloff, *Phys. Rev. A* **80**, 033615 (2009).
- [44] G. Orso, A. Iucci, M. A. Cazalilla, and T. Giamarchi, *Phys. Rev. A* **80**, 033625 (2009).
- [45] G.M. Falco, T.V. Nattermann, and L. Pokrovsky, *Europhys. Lett.* **85**, 30002 (2009).
- [46] L. Pollet, N.V. Prokof'ev, B.V. Svistunov, and M. Troyer, *Phys. Rev. Lett.* **103**, 140402 (2009).
- [47] I.L. Aleiner, B.L. Altshuler, and G.V. Shlyapnikov, *Nature Phys.* **6**, 900 (2010).
- [48] G. Orso, *Phys. Rev. Lett.* **99**, 250402 (2007).
- [49] K. Byczuk, W. Hofstetter, and D. Vollhardt, *Phys. Rev. Lett.* **102**, 146403 (2009).
- [50] K. Byczuk, W. Hofstetter, and D. Vollhardt, *Int. J. Mod. Phys. B* **24**, 1727 (2010).
- [51] Li Han and C.A.R. Sa de Melo, e-print: arXiv.0904.4197.
- [52] A. Sanpera, A. Kantian, L. Sanchez-Palencia, J. Zakrewski, and M. Lewenstein, *Phys. Rev. Lett.* **93**, 040401 (2004).
- [53] V. Ahufinger, L. Sanchez-Palencia, A. Kantian, A. Sanpera, and M. Lewenstein, *Phys. Rev. A* **72**, 063616 (2005).
- [54] J. Wehr, A. Niederberger, L. Sanchez-Palencia, and M. Lewenstein, *Phys. Rev. B* **74**, 224448 (2006).
- [55] A. Niederberger, T. Schulte, J. Wehr, M. Lewenstein, L. Sanchez-Palencia, and K. Sacha, *Phys. Rev. Lett.* **100**, 030403 (2008).
- [56] A. Niederberger, J. Wehr, M. Lewenstein, K. Sacha, *Europhys. Lett.* **86**, 26004 (2009).
- [57] A. Niederberger, M.M. Rams, J. Dziarmaga, F.M. Cucchietti, J. Wehr, and M. Lewenstein, *Phys. Rev. A* **82**, 013630 (2010).
- [58] F. Crépin, G. Zaránd, and P. Simon, *Phys. Rev. Lett.* **105**, 115301 (2010).
- [59] D. Clément, A.F. Varón, M. Hugbart, J.A. Retter, P. Bouyer, L. Sanchez-Palencia, D.M. Gangardt, G.V. Shlyapnikov, and A. Aspect, *Phys. Rev. Lett.* **95**, 170409 (2005).
- [60] C. Fort, L. Fallani, V. Guarrera, J.E. Lye, M. Modugno, D.S. Wiersma, and M. Inguscio, *Phys. Rev. Lett.* **95**, 170410 (2005).
- [61] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, *Nature (London)* **453**, 891 (2008).
- [62] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, *Nature (London)* **453**, 895 (2008).
- [63] B. Deissler, M. Zaccanti, G. Roati, C. D'Errico, M. Fattori, M. Modugno, G. Modugno, and M. Inguscio, *Nature Phys.* **6**, 354 (2010).
- [64] E. Lucioni, B. Deissler, L. Tanzi, G. Roati, M. Zaccanti, M. Modugno, M. Larcher, F. Dalfovo, M. Inguscio, and G. Modugno, *Phys. Rev. Lett.* **106**, 230403 (2011).
- [65] M. White, M. Pasienski, D. McKay, S.Q. Zhou, D. Ceperley, and B. DeMarco, *Phys. Rev. Lett.* **102**, 055301 (2009).
- [66] M. Pasienski, D. McKay, M. White, and B. DeMarco, *Nature Phys.* **6**, 677 (2010).
- [67] J.P. Bouchaud and A. Georges, *Phys. Rep.* **195**, 127 (1990).
- [68] M.C.W. van Rossum and Th.M. Nieuwenhuizen, *Rev. Mod. Phys.* **71**, 313 (1999).
- [69] J. Rammer, *Quantum Transport Theory* (Perseus Books, Reading, Mass., 1998).
- [70] B. Kramer and A. MacKinnon, *Rep. Prog. Phys.* **56**, 1569 (1993).
- [71] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, *Phys. Rev. Lett.* **42**, 673 (1979).
- [72] M. Hartung, T. Wellens, C.A. Müller, K. Richter, and P. Schlagheck, *Phys. Rev. Lett.* **101**, 020603 (2008).
- [73] P. Horak, J.-Y. Courtois, and G. Grynberg, *Phys. Rev. A* **58**, 3953 (1998).
- [74] G. Grynberg, P. Horak, and C. Mennerat-Robilliard, *Europhys. Lett.* **49**, 424 (2000).
- [75] B. Shapiro, *Phys. Rev. Lett.* **99**, 060602 (2007).
- [76] L. Beilin, E. Gurevich, and B. Shapiro, *Phys. Rev. A* **81**, 033612 (2010).
- [77] M. Robert-de-Saint-Vincent, J.-P. Brantut, B. Allard, T. Plisson, L. Pezzé, L. Sanchez-Palencia, A. Aspect, T. Bourdel, and P. Bouyer, *Phys. Rev. Lett.* **104**, 220602 (2010).
- [78] D. Clément, P. Bouyer, A. Aspect, and L. Sanchez-Palencia, *Phys. Rev. A* **77**, 033631 (2008).
- [79] Y.P. Chen, J. Hitchcock, D. Dries, M. Junker, C. Welford, and R.G. Hulet, *Phys. Rev. A* **77**, 033632 (2008).
- [80] D. Dries, S.E. Pollack, J.M. Hitchcock, and R.G. Hulet, *Phys. Rev. A* **82**, 033603 (2010).
- [81] J.W. Goodman, *Speckle Phenomena in Optics* (Roberts and Company Publishers, 2006).
- [82] Note that $c(0) = 1$.
- [83] R. Zallen and H. Scher, *Phys. Rev. B* **4**, 4471 (1971).
- [84] For a review, see M.B. Isichenko, *Rev. Mod. Phys.* **64**, 961 (1992).
- [85] L.N. Smith and C.J. Lobb, *Phys. Rev. B* **20**, 3653 (1979).
- [86] A. Weinrib, *Phys. Rev. B* **26**, 1352 (1982).
- [87] R. Metzler and J. Klafter, *Phys. Rep.* **339**, 1 (2000).
- [88] By numerical construction, the 2D speckle potential is continuous at the box boundaries.
- [89] P. Lugan, A. Aspect, L. Sanchez-Palencia, D. Delande, B. Grémaud, C. A. Müller, and C. Miniatura, *Phys.*

Rev. A **80**, 023605 (2009).

- [90] E. Gurevich and O. Kenneth, Phys. Rev. A **79**, 063617 (2009).
- [91] Alternatively, one can introduce the ballistic path length, $l_r^*(E) \equiv \tau_r^*(E)\sqrt{2E/m}$, which is the characteristic length traveled by a particle, beyond which its trajectory is affected by the disorder.
- [92] I.M. Lifshits, S.A. Gredeskul, and L.A. Pastur, *Introduction to the Theory of Disordered systems* (Wiley, New York, 1988).
- [93] For a red-detuned speckle potential ($V_R < 0$), we find:
- $$n(E_p) = \frac{e^{-E_p/V_R}}{|V_R|} \begin{cases} e^{E/V_R} \Theta(E - E_p) & \text{for } E \leq 0 \\ \Theta(E_p/V_R) & \text{for } E \geq 0. \end{cases}$$
- For $E \leq 0$, the average values of the potential energy and of kinetic energy along each direction are $\langle E_p \rangle = V_R + E$ and $\langle E_k \rangle = |V_R|$, respectively. For $E \geq 0$, we find $\langle E_p \rangle = V_R$ and $\langle E_k \rangle = E + |V_R|$. The latter relations are intuitive in the limit $E \gg |V_R|$. The fact that they hold for $E \geq 0$ is a consequence of phase-space ergodicity in the 2D geometry.
- [94] We remind that, in the numerics, the the particle is initially at the origin, $\mathbf{r} = 0$, with an isotropically-distributed momentum direction. The total energy E is fixed, and the disordered potential is randomly chosen and kept only if $E > V(\mathbf{0})$. The initial potential energy distribution is thus given by $n_0(E_p) \propto P(E_p)\Theta(E - E_p)$. The average initial kinetic energy is $\langle E_k \rangle_0 = E - \langle E_p \rangle_0$, where $\langle E_p \rangle_0$ is the average potential energy. For a blue-detuned speckle potential, an explicit calculation gives $\langle E_k \rangle_0 = V_R(\frac{E/V_R}{1 - e^{-E/V_R}} - 1)$. For a red detuned speckle, $\langle E_k \rangle_0 = |V_R|$ if $E \leq 0$, and $\langle E_k \rangle_0 = E + |V_R|$ if $E \geq 0$. Note that, in the 2D geometry we find that $n_0(E_p)$ equals $n(E_p)$ as calculated in Sec. III C. However, while $n_0(E_p)$ is obtained by averaging over the particle momentum direction, $n(E_p)$ is obtained in the micro-canonical ensemble model, where momentum averaging results from energy distribution.
- [95] In the numerics, we find a slow dependence of the anisotropy factor of the diffusion, D_x/D_y , versus the energy E . However, in the whole energy window that is considered here, we find $D_x/D_y \simeq 3.1$ with an accuracy better than 10%.
- [96] R. Klages, G. Radons, and I.M. Sokolov, *Anomalous Transport* (Wiley-VHC, New York, 2008).
- [97] P. Bathelemy, J. Bertolotti, and D. Wiersma, Nature **453**, 495 (2008).
- [98] N. Mercadier, W. Guerin, M. Chevrollier, and R. Kaiser, Nature Phys. **5**, 602 (2009).
- [99] F. Bardou, J.P. Bouchaud, O. Emilie, A. Aspect, and C. Cohen-Tannoudji, Phys. Rev. Lett. **72**, 203 (1994).
- [100] Similar behavior was identified in dissipative, periodic optical lattices [101]. In the latter case, escape from

trapping regions was induced by dissipative effects. In the present case, the potential is purely conservative and escape from (nearly) trapping regions is due to the topography of the potential near the percolation threshold, which shows thin connecting paths.

- [101] L. Sanchez-Palencia, P. Horak, and G. Grynberg, Eur. Phys. J. D **18**, 353 (2002); L. Sanchez-Palencia and G. Grynberg, Phys. Rev. A **68**, 023404 (2003).
- [102] N. Cherroret and S.E. Skipetrov, Phys. Rev. Lett. **101**, 190406 (2008).
- [103] N. Cherroret and S.E. Skipetrov, Phys. Rev. A **79**, 063604 (2009).
- [104] For a red-detuned speckle potential ($V_R < 0$), Eq. (25) is given by

$$\langle n(E) \rangle = \frac{\beta}{1 + \beta|V_R|} \begin{cases} e^{+E/|V_R|} & \text{for } E \leq 0 \\ e^{-\beta E} & \text{for } E \geq 0, \end{cases}$$

while Eq. (27) is

$$\langle n(E) \rangle = \beta(1 - \beta|V_R|) \begin{cases} e^{-E(\beta - 1/|V_R|)} & \text{for } E \leq 0 \\ e^{-\beta E} & \text{for } E \geq 0. \end{cases}$$

- [105] Explicit calculations give

$$n_{<}(E) = 1 - \frac{e^{-\beta E} - \beta V_R e^{-E/V_R}}{1 - \beta V_R} \quad (36)$$

for the distribution Eq. (26), and

$$n_{<}(E) = 1 - e^{-\beta E} [1 + \beta V_R (1 - e^{-E/V_R})] \quad (37)$$

for the distribution Eq. (28). They correspond to a thermal gas in equilibrium in a bare and a disordered trap, respectively. In the high temperature limit, $\beta V_R \rightarrow 0$, Eqs. (36) and (37) tend to $n_{<}(E) \approx (e^{-E/V_R} + E/V_R - 1)\beta V_R + o(\beta V_R)^2$. In the low temperature limit, $\beta V_R \rightarrow \infty$, Eq. (36) tends to $n_{<}(E) \approx 1 - e^{-E/V_R}$ while Eq. (37) tends to $n_{<}(E) = 1$.

- [106] For an isotropic, ballistic expansion we have $f_E(\mathbf{r}, t; \mathbf{r}_0, t_0) = \Theta(t - t_0) \delta[|\mathbf{r} - \mathbf{r}_0| - (t - t_0)\sqrt{2E/m}]/2\pi(t - t_0)\sqrt{2E/m}$.
- [107] For the distribution of Eq. (26) we have $\bar{E} = k_B T + V_R$. For the parameters of Fig. (14), $k_B T = V_R$, we have $\bar{E} = 2V_R$ and $\tau^*(\bar{E})/\tau_R \sim 10$.
- [108] In the 3D case, the classical equations of motion read

$$\begin{aligned} \frac{d\tilde{\mathbf{r}}}{d\tilde{t}} &= \tilde{\mathbf{p}}, \\ \frac{d\tilde{\mathbf{p}}}{d\tilde{t}} &= -\tilde{\nabla} \left[v_{\text{iso}}(\tilde{x} \sin \theta - \tilde{z} \cos \theta, \tilde{y}) + \frac{(\omega_z \tau_R)^2 \tilde{z}^2}{2} \right], \end{aligned}$$

where the reduced variables are defined in Sec. III A.